Lecture 1

Review of Simple Linear Regression

Reading: ISLR 2021 Chapter 3.1

DSA 8020 Statistical Methods II

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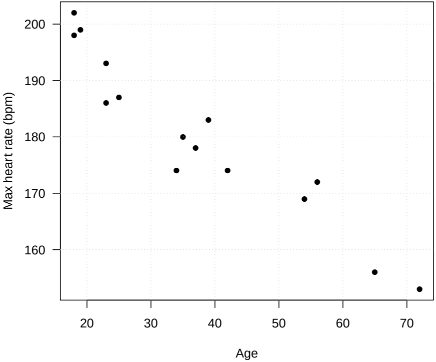
Agenda

1. Simple Linear Regression
2. Parameter Estimation
3. Residual Analysis
4. Confidence/Prediction Intervals
5. Hypothesis Testing

What is Regression Analysis?

Regression analysis: A set of statistical procedures for estimating the relationship between response variable and predictor variable(s)



Simple linear regression: The relationship between the response variable and the predictor variable is approximately linear

Simple Linear Regression (SLR)

y: response variable; x: predictor variable

- In SLR we **assume** there is a **linear relationship** between x and y:

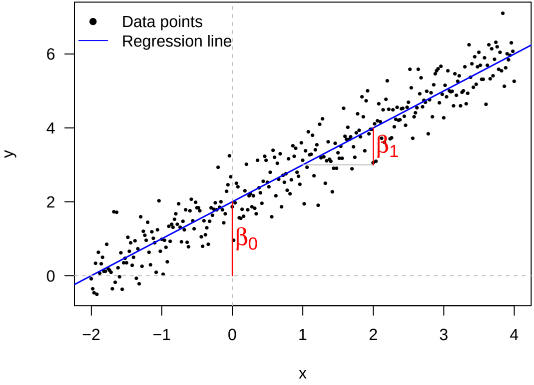
$$y = \beta_0 + \beta_1 x + \varepsilon$$

- We need to estimate β_0 (intercept) and β_1 (slope) based on observed data $\{x_i, y_i\}_{i=1}^n$

- We can use the estimated regression equation to
  - make predictions
  - study the relationship between response and predictor
  - control the response

- Yet we need to quantify our estimation uncertainty regarding the linear relationship

Regression equation: $y = \beta_0 + \beta_1 x$



- β_0: $\mathrm{E}[y]$ when $x = 0$
- β_1: $\mathrm{E}[\Delta y]$ when x increases by 1

Assumptions about the Random Error ε

In order to estimate β_0 and β_1, we make the following assumptions about ε

- $\mathrm{E}[\varepsilon_i] = 0$
- $\mathrm{Var}[\varepsilon_i] = \sigma^2$
- $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Therefore, we have

$$\mathrm{E}[y_i] = \beta_0 + \beta_1 x_i, \text{ and}$$
$$\mathrm{Var}[y_i] = \sigma^2$$

The regression line $\beta_0 + \beta_1 x$ represents the **conditional mean curve** whereas σ^2 measures the magnitude of the **variation** around the regression curve

Estimation: Method of Least Squares

For given observations $\{x_i, y_i\}_{i=1}^n$, choose β_0 and β_1 to minimize the *sum of squared errors*:

$$\ell(\beta_0, \beta_1) = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2$$

Solving the above minimization problem requires some knowledge from Calculus (see notes `LS_SLR.pdf`)

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

We also need to **estimate** σ^2

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2},$$

where $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

Example: Maximum Heart Rate vs. Age

The maximum heart rate `MaxHeartRate` of a person is often said to be related to age `Age` by the equation:

$$\text{MaxHeartRate} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

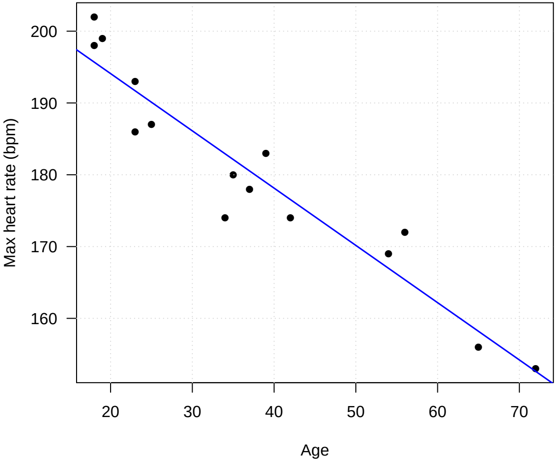
1. Compute the estimates for the regression coefficients
2. Compute the fitted values
3. Compute the estimate for σ

Maximum Heart Rate vs. Age

Output from R (RStudio)

```
> fit <- lm(MaxHeartRate ~ Age)
> summary(fit)

Call:
lm(formula = MaxHeartRate ~ Age)

Residuals:
    Min      1Q  Median      3Q     Max
-8.9258 -2.5383  0.3879  3.1867  6.6242

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 210.04846    2.86694   73.27  < 2e-16 ***
Age          -0.79773    0.06996  -11.40 3.85e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.578 on 13 degrees of freedom
Multiple R-squared:  0.9091,    Adjusted R-squared:  0.9021
F-statistic:   130 on 1 and 13 DF,  p-value: 3.848e-08
```

Assessing Linear Regression Fit



Question: Is linear relationship between max heart rate and age reasonable? ⇒ Residual Analysis

Residuals

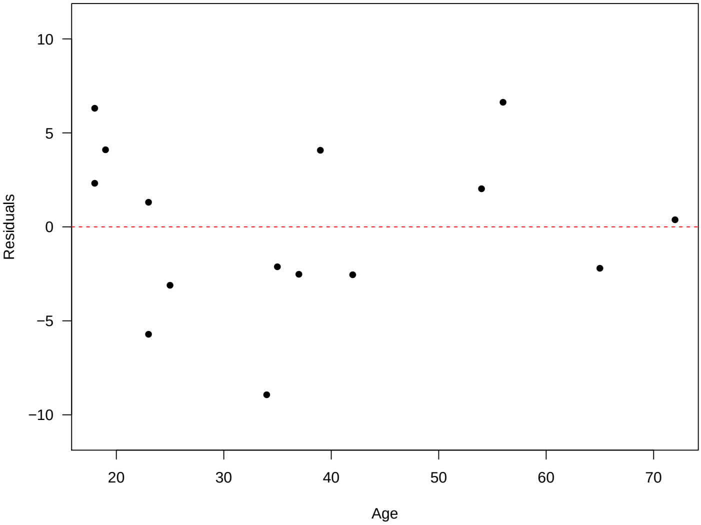
- The residuals are the differences between the observed and fitted values:

$$e_i = y_i - \hat{y}_i,$$

where $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

- Residuals are very useful in assessing the appropriateness of the assumptions on ε_i. Recall
    - $\mathrm{E}[\varepsilon_i] = 0$
    - $\mathrm{Var}[\varepsilon_i] = \sigma^2$
    - $\mathrm{Cov}[\varepsilon_i, \varepsilon_j] = 0, \quad i \neq j$

Residuals Against Predictor Plot



Interpreting Residual Plots

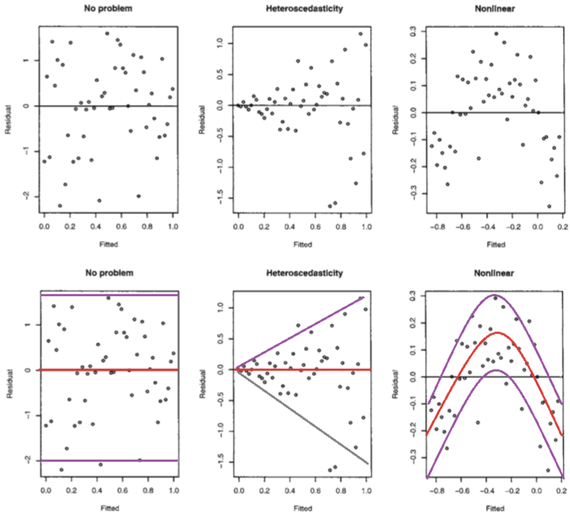
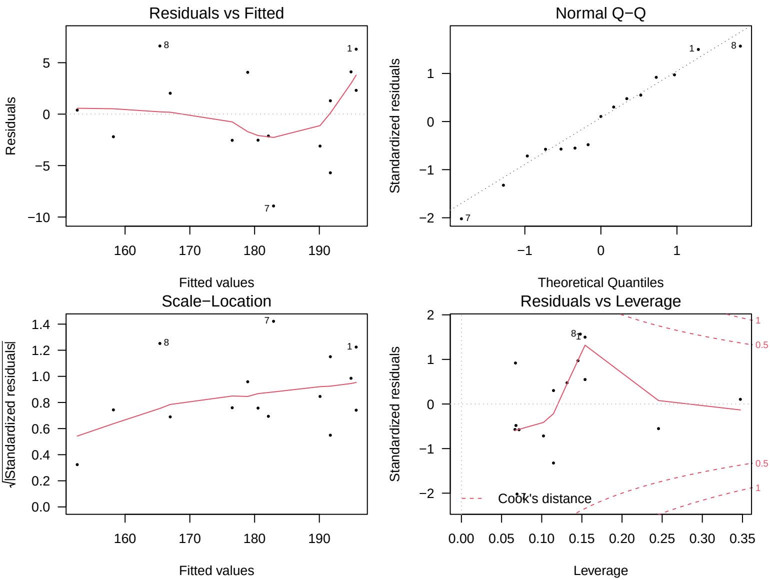
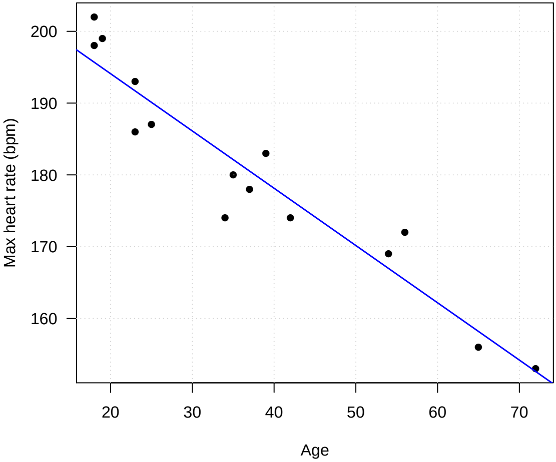


Figure courtesy of Faraway's Linear Models with R (2014, p. 74).

Diagnostic Plots in R



How (Un)certain We Are?



Can we formally quantify our estimation uncertainty? $\Rightarrow$

We need additional (distributional) assumption on ε

Normal Error Regression Model

Recall

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

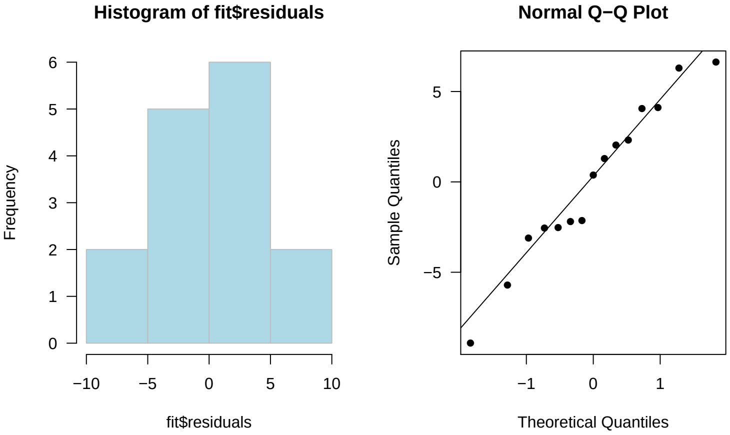
- Further assume $\varepsilon_i \sim \mathrm{N}(0, \sigma^2) \Rightarrow y_i | x_i \sim \mathrm{N}(\beta_0 + \beta_1 x_i, \sigma^2)$
- With normality assumption, we can derive the **sampling distribution** of $\hat{\beta}_1$ and $\hat{\beta}_0 \Rightarrow$

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{SE}(\hat{\beta}_1)} \sim t_{n-2}, \quad \hat{SE}(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}}$$

$$\frac{\hat{\beta}_0 - \beta_0}{\hat{SE}(\hat{\beta}_0)} \sim t_{n-2}, \quad \hat{SE}(\hat{\beta}_0) = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right)}$$

where t_{n-2} denotes the Student's t distribution with $n-2$ degrees of freedom

Assessing Normality Assumption on ε



The Q-Q plot is more effective in detecting subtle departures from normality, especially in the tails.

Confidence Intervals for β_0 and β_1

- Recall $\frac{\hat{\beta}_1 - \beta_1}{\hat{SE}(\hat{\beta}_1)} \sim t_{n-2}$, we use this fact to construct a **confidence interval (CI)** for β_1:

$$\left[\hat{\beta}_1 - t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_1), \hat{\beta}_1 + t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_1)\right],$$

  where α is the **confidence level** and $t_{\alpha/2,n-2}$ denotes the $1-\alpha/2$ percentile of a student's t distribution with $n-2$ degrees of freedom

- Similarly, we can construct a CI for β_0:

$$\left[\hat{\beta}_0 - t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_0), \hat{\beta}_0 + t_{\alpha/2,n-2}\hat{SE}(\hat{\beta}_0)\right]$$

Confidence Interval of $\mathrm{E}(y_{new})$

- We often interested in estimating the **mean** response for an unobserved predictor value, say, x_{new}. Therefore we would like to construct CI for $\mathrm{E}[y_{new}]$, the corresponding **mean response**

- We need sampling distribution of $\widehat{\mathrm{E}(y_{new})}$ to form CI:
  - $\frac{\widehat{\mathrm{E}(y_{new})}-\mathrm{E}(y_{new})}{\hat{SE}(\widehat{\mathrm{E}(y_{new})})} \sim t_{n-2}, \quad \hat{SE}(\widehat{\mathrm{E}(y_{new})}) = \hat{\sigma}\sqrt{\left(\frac{1}{n} + \frac{(x_{new}-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)}$
  - CI:

$$\left[\hat{y}_{new} - t_{\alpha/2,n-2}\hat{SE}(\widehat{\mathrm{E}(y_{new})}), \hat{y}_{new} + t_{\alpha/2,n-2}\hat{SE}(\widehat{\mathrm{E}(y_{new})})\right]$$

- **Quiz:** Use this formula to construct CI for β_0

Prediction Interval of y_{new}

- Suppose we want to predict the response of a future observation y_{new} given $x = x_{new}$
- We need to account for added variability as a new observation does not fall directly on the regression line (i.e., $y_{new} = \mathrm{E}[y_{new}] + \varepsilon_{new}$)
- Replace $\hat{SE}(\widehat{\mathrm{E}(y_{new})})$ by $\hat{SE}(\hat{y}_{new}) = \hat{\sigma}\sqrt{\left(1 + \frac{1}{n} + \frac{(x_{new}-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)}$ to construct CIs for Y_{new}

Maximum Heart Rate vs. Age Revisited

The maximum heart rate `MaxHeartRate` (HR_{max}) of a person is often said to be related to age `Age` by the equation:

$$\text{HR}_{max} = 220 - \text{Age}.$$

Suppose we have 15 people of varying ages are tested for their maximum heart rate (bpm)

Age	18	23	25	35	65	54	34	56	72	19	23	42	18	39	37
HR_{max}	202	186	187	180	156	169	174	172	153	199	193	174	198	183	178

- Construct the 95% CI for β_1
- Compute the estimate for mean `MaxHeartRate` given `Age` $= 40$ and construct the associated 90% CI
- Construct the prediction interval for a new observation given `Age` $= 40$

Maximum Heart Rate vs. Age: Hypothesis Test for Slope

1. $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_1 - 0}{\hat{SE}(\hat{\beta}_1)} = \frac{-0.7977}{0.06996} = -11.40$
3. Compute **P-value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) = 3.85 \times 10^{-8}$
4. Compare to α and draw conclusion:

> Reject H_0 at $\alpha = .05$ level, evidence suggests a negative linear relationship between `MaxHeartRate` and `Age`

Maximum Heart Rate vs. Age: Hypothesis Test for Intercept

1. $H_0 : \beta_0 = 0$ vs. $H_a : \beta_0 \neq 0$
2. Compute the **test statistic**: $t^* = \frac{\hat{\beta}_0 - 0}{\hat{SE}(\hat{\beta}_0)} = \frac{210.0485}{2.86694} = 73.27$
3. Compute **P-value**: $\mathrm{P}(|t^*| \geq |t_{obs}|) \simeq 0$
4. Compare to α and draw conclusion:

> Reject H_0 at $\alpha = .05$ level, evidence suggests evidence suggests the intercept (the expected `MaxHeartRate` at age 0) is different from 0

Summary

In this lecture, we reviewed

- Simple Linear Regression: $y = \beta_0 + \beta_1 x + \varepsilon,\ \varepsilon \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$
- Method of Least Squares for parameter estimation

$$\hat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}=(\beta_0,\beta_1)}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

- Residual analysis to check model assumptions
- Confidence/Prediction Intervals and Hypothesis Testing

R Funcations

- Fitting linear models

  `object <- lm(formula, data)` where the formula is specified via `y ~ x` $\Rightarrow y$ is modeled as a linear function of x

- Diagnostic plots

  ```
  plot(object)
  ```

- Summarizing fits

  ```
  summary(object)
  ```

- Making predictions

  ```
  predict(object, newdata)
  ```

- Confidence Intervals for Model Parameters

  ```
  confint(object)
  ```