# Lecture 3

## Multiple Linear Regression I

Reading: Forecasting, Time Series, and Regression (4th edition) by Bowerman, O'Connell, and Koehler: Chapter 4

*MATH 4070: Regression and Time-Series Analysis*

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Notes

### Agenda

1. **Multiple Linear Regression**
2. **Estimation & Inference**
3. **Assessing Model Fit**





Notes

### Multiple Regression Analysis

**Goal**: To model the population relationship between two or more predictors ($\mathbf{X}$'s) and a response ($Y$).

$$\text{Model: } Y = f(\mathbf{x}) + \varepsilon.$$

**Example**: Species diversity on the Galapagos Islands.
We are interested in studying the relationship between the number of plant species (`Species`) and the following geographic variables: `Area, Elevation, Nearest, Scruz, Adjacent`.





Notes

## Data: Species Diversity on the Galapagos Islands

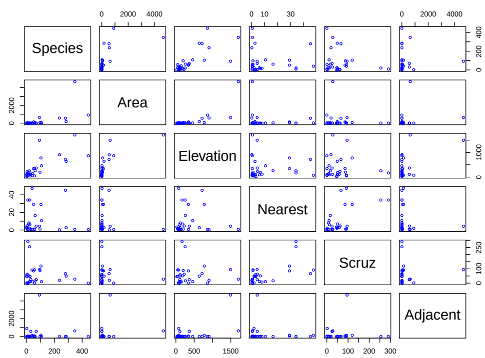
| | Species | Endemics | Area | Elevation | Nearest | Scruz | Adjacent |
|---|---|---|---|---|---|---|---|
| Baltra | 58 | 23 | 25.09 | 346 | 0.6 | 0.6 | 1.84 |
| Bartolome | 31 | 21 | 1.24 | 109 | 0.6 | 26.3 | 572.33 |
| Caldwell | 3 | 3 | 0.21 | 114 | 2.8 | 58.7 | 0.78 |
| Champion | 25 | 9 | 0.10 | 46 | 1.9 | 47.4 | 0.18 |
| Coamano | 2 | 1 | 0.05 | 77 | 1.9 | 1.9 | 903.82 |
| Daphne.Major | 18 | 11 | 0.34 | 119 | 8.0 | 8.0 | 1.84 |
| Daphne.Minor | 24 | 0 | 0.08 | 93 | 6.0 | 12.0 | 0.34 |
| Darwin | 10 | 7 | 2.33 | 168 | 34.1 | 290.2 | 2.85 |
| Eden | 8 | 4 | 0.03 | 71 | 0.4 | 0.4 | 17.95 |
| Enderby | 2 | 2 | 0.18 | 112 | 2.6 | 50.2 | 0.10 |
| Espanola | 97 | 26 | 58.27 | 198 | 1.1 | 88.3 | 0.57 |
| Fernandina | 93 | 35 | 634.49 | 1494 | 4.3 | 95.3 | 4669.32 |
| Gardner1 | 58 | 17 | 0.57 | 49 | 1.1 | 93.1 | 58.27 |
| Gardner2 | 5 | 4 | 0.78 | 227 | 4.6 | 62.2 | 0.21 |
| Genovesa | 40 | 19 | 17.35 | 76 | 47.4 | 92.2 | 129.49 |
| Isabela | 347 | 89 | 4669.32 | 1707 | 0.7 | 28.1 | 634.49 |
| Marchena | 51 | 23 | 129.49 | 343 | 29.1 | 85.9 | 59.56 |
| Onslow | 2 | 2 | 0.01 | 25 | 3.3 | 45.9 | 0.10 |
| Pinta | 104 | 37 | 59.56 | 777 | 29.1 | 119.6 | 129.49 |
| Pinzon | 108 | 33 | 17.95 | 458 | 10.7 | 10.7 | 0.03 |
| Las.Plazas | 12 | 9 | 0.23 | 94 | 0.5 | 0.6 | 25.09 |
| Rabida | 70 | 30 | 4.89 | 367 | 4.4 | 24.4 | 572.33 |
| SanCristobal | 280 | 65 | 551.62 | 716 | 45.2 | 66.6 | 0.57 |
| SanSalvador | 237 | 81 | 572.33 | 906 | 0.2 | 19.8 | 4.89 |
| SantaCruz | 444 | 95 | 903.82 | 864 | 0.6 | 0.0 | 0.52 |
| SantaFe | 62 | 28 | 24.08 | 259 | 16.5 | 16.5 | 0.52 |
| SantaMaria | 285 | 73 | 170.92 | 640 | 2.6 | 49.2 | 0.10 |
| Seymour | 44 | 16 | 1.84 | 147 | 0.6 | 9.6 | 25.09 |
| Tortuga | 16 | 8 | 1.24 | 186 | 6.8 | 50.9 | 17.95 |
| Wolf | 21 | 12 | 2.85 | 253 | 34.1 | 254.7 | 2.33 |





Notes

## How Do Geographic Variables Affect Species Diversity?








Notes

## Let's Take a Look at the Correlation Matrix

Here we compute the correlation coefficients between the response (`Species`) and predictors (all the geographic variables)

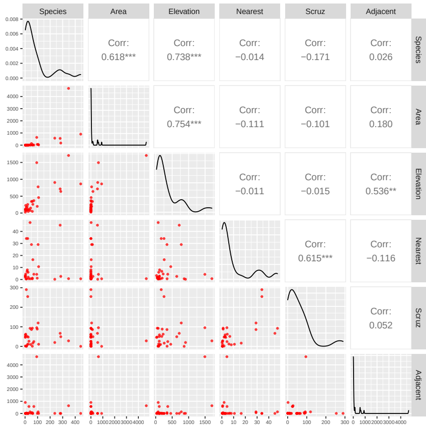
```
> round(cor(gala[, -2]), 3)
          Species   Area Elevation Nearest  Scruz Adjacent
Species     1.000  0.618     0.738  -0.014 -0.171    0.026
Area        0.618  1.000     0.754  -0.111 -0.101    0.180
Elevation   0.738  0.754     1.000  -0.011 -0.015    0.536
Nearest    -0.014 -0.111    -0.011   1.000  0.615   -0.116
Scruz      -0.171 -0.101    -0.015   0.615  1.000    0.052
Adjacent    0.026  0.180     0.536  -0.116  0.052    1.000
```





Notes

## Combining Two Pieces of Information in One Plot








Notes

## Multiple Linear Regression Model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_{p-1} X_{p-1} + \varepsilon$$

- The above relationship holds for every individual in the population, and $\mathbb{E}(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon) = \sigma^2$
- The population of individual error terms ($\varepsilon$'s) follows normal distribution
- Observations are independent (true if individuals are selected randomly)

$$\Rightarrow \varepsilon \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2)$$





Notes

## Model 1: `Species` ~ `Elevation`

```
Call:
lm(formula = Species ~ Elevation, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max 
-218.319  -30.721  -14.690    4.634  259.180 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 11.33511   19.20529   0.590     0.56    
Elevation    0.20079    0.03465   5.795 3.18e-06 ***
---
Signif. codes:  
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 78.66 on 28 degrees of freedom
Multiple R-squared:  0.5454,    Adjusted R-squared:  0.5291 
F-statistic: 33.59 on 1 and 28 DF,  p-value: 3.177e-06
```

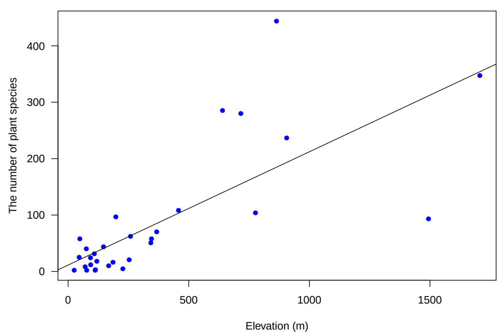




Notes

## Model 1 Fit

$$\widehat{\text{Species}} = 11.33511 + 0.20079 \times \text{Elevation},$$
$$\hat{\sigma} = 78.66,\ \mathrm{R}^2 = 0.5454$$






Notes

## Model 2: `Species ~ Elevation + Area`

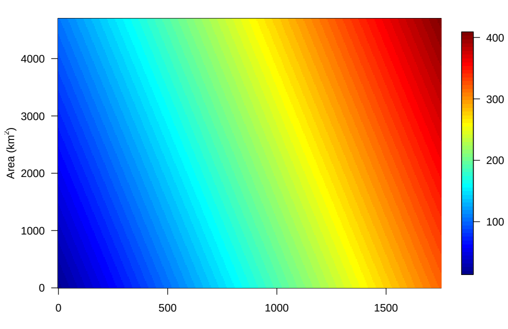
```
Call:
lm(formula = Species ~ Elevation + Area, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max 
-192.619  -33.534  -19.199    7.541  261.514 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)   
(Intercept) 17.10519   20.94211   0.817  0.42120   
Elevation    0.17174    0.05317   3.230  0.00325 **
Area         0.01880    0.02594   0.725  0.47478   
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 79.34 on 27 degrees of freedom
Multiple R-squared:  0.554,	Adjusted R-squared:  0.521 
F-statistic: 16.77 on 2 and 27 DF,  p-value: 1.843e-05
```



Notes

## Model 2 Fit

$$\widehat{\text{Species}} = 17.10519 + 0.17174 \times \text{Elevation} + 0.01880 \times \text{Area},$$
$$\hat{\sigma} = 79.34,\ \mathrm{R}^2 = 0.554$$






Notes

## Model 3: $\texttt{Species} \sim \texttt{Elevation} + \texttt{Area} + \texttt{Adjacent}$

```
Call:
lm(formula = Species ~ Elevation + Area + Adjacent, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max
-124.064  -34.283   -8.733   27.972  195.973

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -5.71893   16.90706  -0.338  0.73789
Elevation    0.31498    0.05211   6.044  2.2e-06 ***
Area        -0.02031    0.02181  -0.931  0.36034
Adjacent    -0.07528    0.01698  -4.434  0.00015 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 61.01 on 26 degrees of freedom
Multiple R-squared:  0.746,     Adjusted R-squared:  0.7167
F-statistic: 25.46 on 3 and 26 DF,  p-value: 6.683e-08
```





Notes

## “Full Model”

```
lm(formula = Species ~ Area + Elevation + Nearest + Scruz + Adjacent,
    data = gala)

Residuals:
     Min       1Q   Median       3Q      Max
-111.679  -34.898   -7.862   33.460  182.584

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  7.068221  19.154198   0.369 0.715351
Area        -0.023938   0.022422  -1.068 0.296318
Elevation    0.319465   0.053663   5.953 3.82e-06
Nearest      0.009144   1.054136   0.009 0.993151
Scruz       -0.240524   0.215402  -1.117 0.275208
Adjacent    -0.074805   0.017700  -4.226 0.000297

(Intercept)
Area
Elevation   ***
Nearest
Scruz
Adjacent    ***
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 60.98 on 24 degrees of freedom
Multiple R-squared:  0.7658,    Adjusted R-squared:  0.7171
F-statistic:  15.7 on 5 and 24 DF,  p-value: 6.838e-07
```





Notes

## MLR Topics

Similar to SLR, we will discuss

- Estimation
- Inference
- Diagnostics and Remedies

We will also discuss some new topics

- Model Selection
- Multicollinearity





Notes

## Multiple Linear Regression in Matrix Notation

Given the actual data, we can write MLR model as:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{2,1} & \cdots & x_{p-1,1} \\ 1 & x_{1,2} & x_{2,2} & \cdots & x_{p-1,2} \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 1 & x_{1,n} & x_{2,n} & \cdots & x_{p-1,n} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

It will be more convenient to put this in a matrix representation as:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Error Sum of Squares (SSE)

$= \sum_{i=1}^{n}\left(y_i - \left(\beta_0 + \sum_{j=1}^{p-1}\beta_j x_{j,i}\right)\right)^2$ can be expressed as:

$$(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})$$

> Next, we are going to find $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_{p-1})$ to minimize SSE as our estimate for $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_{p-1})$





Notes

## Estimating Regression Coefficients

We apply method of least squares to minimize $(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})$ to obtain $\hat{\boldsymbol{\beta}}$

What is important is the **orthogonality**, which leads to the following:

- $\sum_i^n (y_i - \hat{y}_i) = 0$
- $\sum_i^n (y_i - \hat{y}_i)x_{1,i} = 0$
- $\vdots$
- $\sum_i^n (y_i - \hat{y}_i)x_{p-1,i} = 0$

**Note**: The first equation states that the mean of the residuals is 0, while the other equations indicate that the residuals are uncorrelated with the independent variables

The resulting least squares estimate is

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y}$$

(see `LS_MLR.pdf` for the derivation)





Notes

## Estimation of $\sigma^2$

- Fitted values:

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{y} = \boldsymbol{H}\boldsymbol{y}$$

- Residuals:

$$\boldsymbol{e} = \boldsymbol{y} - \hat{\boldsymbol{y}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{y}$$

- Similar as we did in SLR

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\boldsymbol{e}^T\boldsymbol{e}}{n-p} \\ &= \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})^T(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-p} \\ &= \frac{\text{SSE}}{n-p} \\ &= \text{MSE} \end{aligned}$$

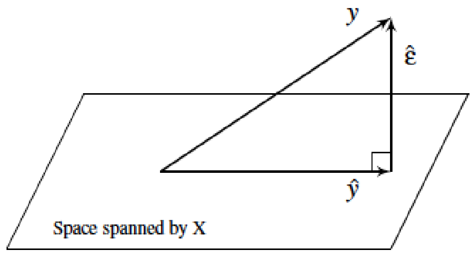




Notes

## Geometric Representation of Least Squares Estimation

Projecting the observed response $y$ into a space spanned by $X$




Source: Linear Model with R 2nd Ed, Faraway, p. 15



Notes

## Regression with Numerical and Categorical Predictors

What if some of the predictors are categorical variables?

**Example**: Salaries for Professors Data Set

```
> head(Salaries)
       rank discipline yrs.since.phd yrs.service  sex salary
1      Prof          B            19          18 Male 139750
2      Prof          B            20          16 Male 173200
3  AsstProf          B             4           3 Male  79750
4      Prof          B            45          39 Male 115000
5      Prof          B            40          41 Male 141500
6 AssocProf          B             6           6 Male  97000
```

We have three categorical variables, namely, `rank`, `discipline`, and `sex`.

⇒ We will need to create **dummy (indicator) variables** for those categorical variables



Notes

## Dummy Variable

For binary categorical variables:

$$x_{\texttt{sex}} = \begin{cases} 1 & \text{if } \texttt{sex} = \text{male}, \\ 0 & \text{if } \texttt{sex} = \text{female}. \end{cases}$$

$$x_{\texttt{discip}} = \begin{cases} 0 & \text{if } \texttt{discip} = \text{A}, \\ 1 & \text{if } \texttt{discip} = \text{B}. \end{cases}$$

For categorical variable with more than two categories:

$$x_{\texttt{rank1}} = \begin{cases} 0 & \text{if } \texttt{rank} = \text{Assistant Prof}, \\ 1 & \text{if } \texttt{rank} = \text{Associated Prof}. \end{cases}$$

$$x_{\texttt{rank2}} = \begin{cases} 0 & \text{if } \texttt{rank} = \text{Associated Prof}, \\ 1 & \text{if } \texttt{rank} = \text{Full Prof}. \end{cases}$$



Notes

## Design Matrix

```
> head(X)
  (Intercept) rankAssocProf rankProf disciplineB yrs.since.phd
1           1             0        1           1            19
2           1             0        1           1            20
3           1             0        0           1             4
4           1             0        1           1            45
5           1             0        1           1            40
6           1             1        0           1             6
  yrs.service sexMale
1          18       1
2          16       1
3           3       1
4          39       1
5          41       1
6           6       1
```

With the design matrix $\boldsymbol{X}$, we can now use method of least squares to fit the model $\boldsymbol{Y} = \boldsymbol{X}\beta + \varepsilon$





Notes

## Model Fit: $\texttt{lm(salary} \sim \texttt{rank} + \texttt{sex} + \texttt{discipline} + \texttt{yrs.since.phd)}$

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)   67884.32    4536.89  14.963  < 2e-16 ***
disciplineB   13937.47    2346.53   5.940 6.32e-09 ***
rankAssocProf 13104.15    4167.31   3.145  0.00179 ** 
rankProf      46032.55    4240.12  10.856  < 2e-16 ***
sexMale        4349.37    3875.39   1.122  0.26242    
yrs.since.phd    61.01     127.01   0.480  0.63124    
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 22660 on 391 degrees of freedom
Multiple R-squared:  0.4472,	Adjusted R-squared:  0.4401 
F-statistic: 63.27 on 5 and 391 DF,  p-value: < 2.2e-16
```

**Question**: Interpretation of the slopes of these dummy variables (e.g. $\hat{\beta}_{\texttt{rankAssocProf}}$)? Interpretation of the intercept?

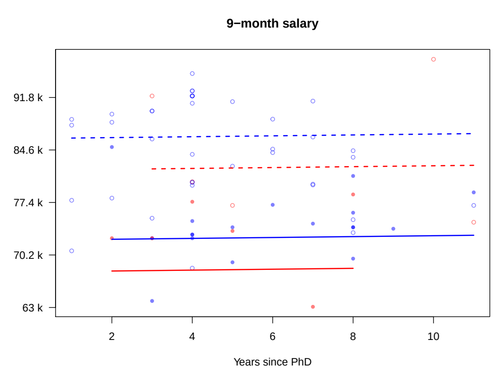




Notes

## Model Fit for Assistant Professors

| Color | Line Type |
|---|---|
| Red: Female | —–: Applied (discipline B) |
| Blue: Male | - - -: Theoretical (discipline A) |








Notes

## Other Type of Predictor Variables: Polynomial regression

Suppose we would like to model the relationship between response $Y$ and a predictor $x$ as a $p_{\text{th}}$ degree polynomial in $x$:

$$Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_p x^p + \varepsilon$$

Polynomial regression can be treated as a special case of multiple linear regression, with the design matrix taking the following form:

$$\boldsymbol{X} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^p \\ 1 & x_2 & x_2^2 & \cdots & x_2^p \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^p \end{pmatrix}$$

One can also include the interaction terms; for example:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + \beta_5 x_1 x_2 + \varepsilon$$





Notes

## Transformed Response Variables

Consider the following models:

$$\log(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon;$$
$$Y = \frac{1}{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon},$$

both of which can be expressed as follws

$$Y^* = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon;$$
$$Y^{**} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon,$$

respectively, where $Y^* = \log(Y)$, and $Y^{**} = 1/Y$.





Notes

## Analysis of Variance (ANOVA) Approach to Regression

**Partitioning Sums of Squares**

- Total sums of squares in response

$$\text{SST} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- We can rewrite SST as

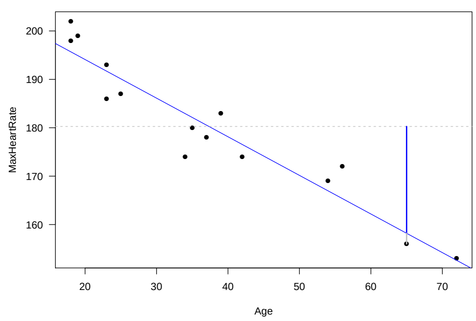
$$\begin{aligned}\sum_{i=1}^{n} (y_i - \bar{y})^2 &= \sum_{i=1}^{n} (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2 \\ &= \underbrace{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}_{\text{“Error”: SSE}} + \underbrace{\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2}_{\text{Model: SSR}}\end{aligned}$$





Notes

## Partitioning Total Sums of Squares: A Graphical Illustration








Notes

## ANOVA Table & $F$-Test

To answer the question: Is **at least** one of the predictors $x_1, \cdots, x_{p-1}$ useful in predicting the response $y$?

| **Source** | df | SS | MS | $F$-Value |
|---|---|---|---|---|
| Model | $p-1$ | SSR | MSR = SSR/$(p-1)$ | MSR/MSE |
| Error | $n-p$ | SSE | MSE = SSE/$(n-p)$ | |
| Total | $n-1$ | SST | | |

- $F$-test: Tests if the predictors $\{x_1, \cdots, x_{p-1}\}$ collectively help explain the variation in $y$
  - $H_0: \beta_1 = \beta_2 = \cdots = \beta_{p-1} = 0$
  - $H_a:$ at least one $\beta_k \neq 0, \quad 1 \leq k \leq p-1$
  - $F^* = \frac{\text{MSR}}{\text{MSE}} = \frac{\text{SSR}/(p-1)}{\text{SSE}/(n-p)} \overset{H_0}{\sim} F_{p-1,n-p}$
  - Reject $H_0$ if $F^* > F_{1-\alpha,p-1,n-p}$





Notes

## Testing Individual Predictor

- We can show that $\hat{\boldsymbol{\beta}} \sim \text{N}_p\left(\boldsymbol{\beta}, \sigma^2\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right) \Rightarrow \hat{\beta}_k \sim \text{N}(\beta_k, \sigma^2_{\hat{\beta}_k})$
- Perform $t$-Test:
  - $H_0: \beta_k = 0$ vs. $H_a: \beta_k \neq 0$
  - $\frac{\hat{\beta}_k - \beta_k}{\hat{\text{se}}(\hat{\beta}_k)} \sim t_{n-p} \Rightarrow t^* = \frac{\hat{\beta}_k}{\hat{\text{se}}(\hat{\beta}_k)} \overset{H_0}{\sim} t_{n-p}$
  - Reject $H_0$ if $|t^*| > t_{1-\alpha/2,n-p}$
- Confidence interval for $\beta_k$:

$$\hat{\beta}_k \pm t_{1-\alpha/2,n-p}\hat{\text{se}}(\hat{\beta}_k)$$



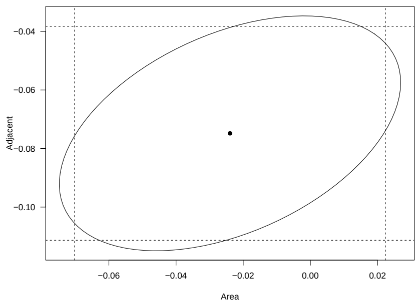


Notes

## Confidence Intervals and Confidence Ellipsoids

Comparing with individual confidence interval, confidence ellipsoids can provide additional information when inference with multiple parameters is of interest. A $100(1-\alpha)$% confidence ellipsoid for $\boldsymbol{\beta}$ can be constructed using:

$$(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T \boldsymbol{X}^T \boldsymbol{X} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \leq p\hat{\sigma}^2 F^{\alpha}_{p,n-p}.$$








Notes

## Quantifying Model Fit using Coefficient of Determination $R^2$

- **Coefficient of determination** $R^2$ describes proportional of the variance in the response variable that is predictable from the predictors

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}, \quad 0 \leq R^2 \leq 1$$

- $R^2$ increases with the increasing $p$, the number of the predictors
  - Adjusted $R^2$, denoted by $R^2_{\text{adj}} = 1 - \frac{\text{SSE}/(n-p)}{\text{SST}/(n-1)}$ attempts to account for $p$





Notes

## $R^2$ vs. $R^2_{\text{adj}}$ Example

Suppose the true relationship between response $Y$ and predictors $(x_1, x_2)$ is

$$y = 5 + 2x_1 + \varepsilon,$$

where $\varepsilon \sim \text{N}(0,1)$ and $x_1$ and $x_2$ are independent to each other. Let's fit the following two models to the "data"

Model 1: $Y = \beta_0 + \beta_1 x_1 + \varepsilon^1$
Model 2: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon^2$

**Question:** Which model will "win" in terms of $R^2$?

Let's conduct a Monte Carlo simulation to study this





Notes

## Outline of Monte Carlo Simulation

1. Generating a large number (e.g., $M = 500$) of "data sets", where each has exactly the same $\{x_{1,i}, x_{2,i}\}_{i=1}^n$ but different values of response $\{y_i = 5 + 2x_{1,i} + \varepsilon_i\}_{i=1}^n$
2. Fitting model 1: $y = \beta_0 + \beta_1 x_1 + \varepsilon^1$ (true model) and model 2: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon^2$, respectively for each simulating data set and calculating their $R^2$ and $R^2_{adj}$
3. Summarizing $\{R^2_j\}_{j=1}^M$ and $\{R^2_{adj,j}\}_{j=1}^M$ for model 1 and model 2





Notes

## An Example of Model 1 Fit

```
> summary(fit1)

Call:
lm(formula = y ~ x1)

Residuals:
    Min      1Q  Median      3Q     Max
-1.6085 -0.5056 -0.2152  0.6932  2.0118

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.1720     0.1534   33.71  < 2e-16 ***
x1            1.8660     0.1589   11.74 2.47e-12 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8393 on 28 degrees of freedom
Multiple R-squared:  0.8313,    Adjusted R-squared:  0.8253
F-statistic:   138 on 1 and 28 DF,  p-value: 2.467e-12
```





Notes

## An Example of Model 2 Fit

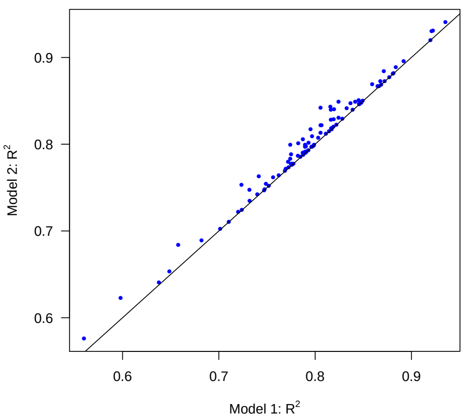
```
> summary(fit2)

Call:
lm(formula = y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-1.3926 -0.5775 -0.1383  0.5229  1.8385

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.1792     0.1518  34.109  < 2e-16 ***
x1            1.8994     0.1593  11.923 2.88e-12 ***
x2           -0.2289     0.1797  -1.274    0.213
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8301 on 27 degrees of freedom
Multiple R-squared:  0.8408,    Adjusted R-squared:  0.8291
F-statistic: 71.32 on 2 and 27 DF,  p-value: 1.677e-11
```





Notes

## $R^2$: Model 1 vs. Model 2

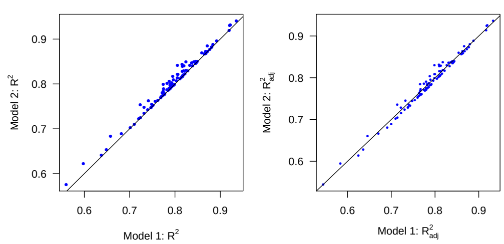






Notes

## $R^2_{adj}$: Model 1 vs. Model 2




**Takeaways**:

- $R^2$ always pick the more "complex" model (i.e., with more predictors), even the simpler model is the true model
- $R^2_{adj}$ has a better chance to pick the "right" model




Notes

## Summary

These slides cover:

- Multiple Linear Regression: Model and Parameter Estimation
- Inference: $F$-test and $t$-test; Confidence intervals/ellipsoids
- Assessing Model Fit: $R^2$ and $R^2_{\text{adj}}$
- Monte Carlo Simulation

`R` functions to know:

- `image.plot` in the `fields` library and `scatter3D` in the `plot3D` library for visualization
- `anova` for computing the ANOVA table




Notes