Lecture 6

Autocorrelation and Time Series Models

Reading: Forecasting, Time Series, and Regression (4th edition) by Bowerman, O'Connell, and Koehler: Chapter 6

MATH 4070: Regression and Time-Series Analysis

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Objectives of Time Series Analysis
Time Series Models
Mean and Autocovaraince Functions
Stationarity

Notes

Agenda

1. Objectives of Time Series Analysis
2. Time Series Models
3. Mean and Autocovaraince Functions
4. Stationarity

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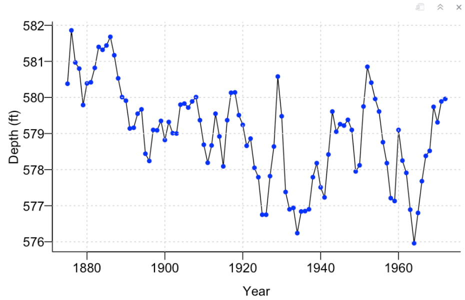
Level of Lake Huron 1875–1972

```{r}
par(mar = c(3.2, 3.2, 0.5, 0.5), mgp = c(2, 0.5, 0), bty = "L")
data(LakeHuron)
plot(LakeHuron, ylab = "Depth (ft)", xlab = "Year", las = 1)
points(LakeHuron, cex = 0.8, col = "blue", pch = 16)
grid()
```



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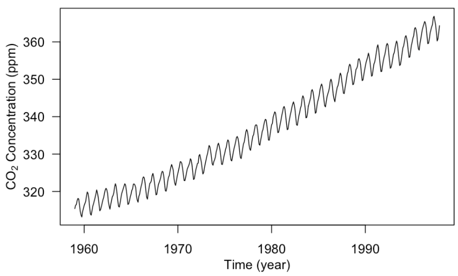
Mauna Loa Atmospheric CO_2 Concentration

```{r}
data(co2)
par(mar = c(3.8, 4, 0.8, 0.6))
plot(co2, las = 1, xlab = "", ylab = "")
mtext("Time (year)", side = 1, line = 2)
mtext(expression(paste("CO"[2], " Concentration (ppm)")), side = 2, line = 2.5)
```



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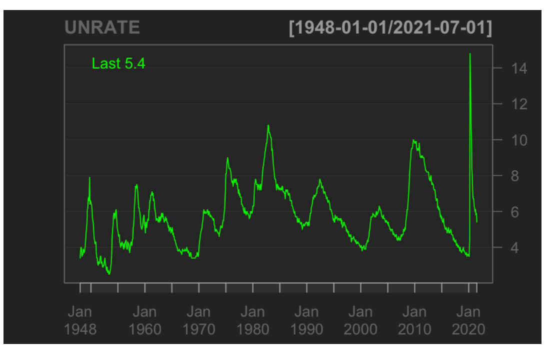
US Unemployment Rate 1948 Jan. – 2021 July

[Source: St. Louis Federal Reserve Bank's FRED system]



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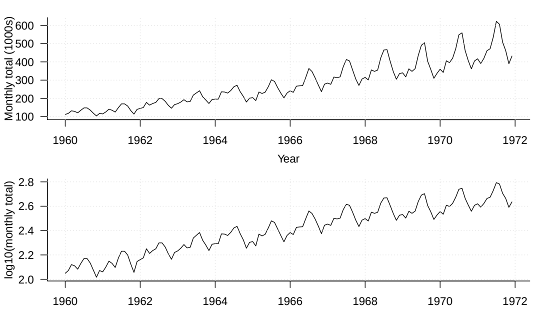
Airline Passengers Example

The data set `airpassengers`, which are the monthly totals of international airline passengers from 1960 to 1971.



Here we stabilize the variance with a $\log_{10}$ transformation

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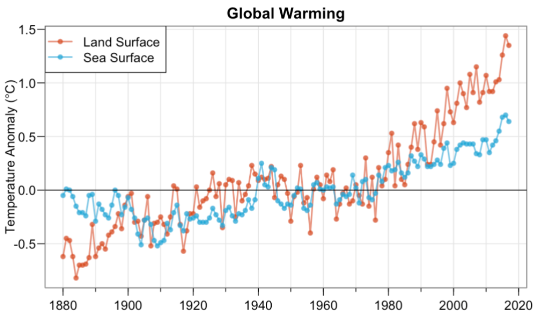
Global Annual Temperature Anomalies

[Source: NASA GISS Surface Temperature Analysis]



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A Simulated Time Series

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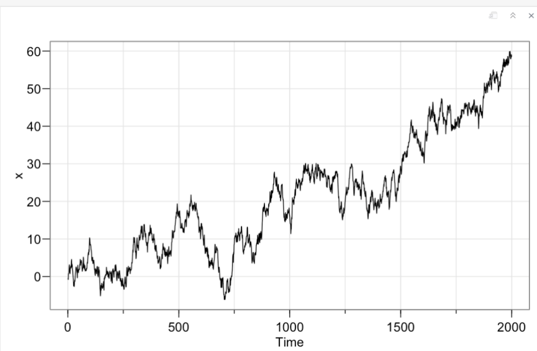
```{r}
set.seed(123)
w <- rnorm(2000); x <- cumsum(w); tsplot(x, las = 1)
```



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Objectives of Time Series Analysis

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Some Objectives of Time Series Analysis

Statistical Modeling: Find a statistical model that adequately explains the observed time series

- For example, identify a model which can account for the fact that the depths of Lake Huron are correlated with differ years and with a decreasing long-term trend
- The fitted model can be used for further statistical inference, for instant, to answer the question like: Is there evidence of decreasing trend in the Lake Huron depths?

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Some Objectives of Time Series Analysis, Cont'd

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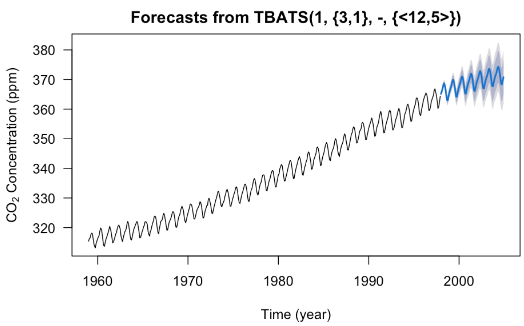
Forecasting is perhaps the most common objective. One observe a time series of given length and wish to **predict** or **forecast** future values of the time series based on those already observed.



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Some Objectives of Time Series Analysis, Cont'd

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- **Adjustment**: an example would be seasonal adjustment, where the seasonal component is estimated and then removed to better understand the underlying trend
- **Simulation**: use a time series model (which adequately describes a physical process) as a surrogate to *simulate repeatedly in order to approximate how the physical process behaves*
- **Control**: adjust various input (control) parameters to make the time series fit more closely to a given standard (many examples from statistical quality control)

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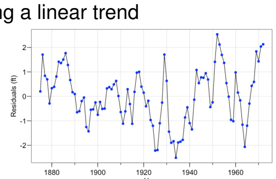
Time Series Models

Notes

Lake Huron Time Series

- Time series analysis is the area of statistics which deals with the analysis of dependency between different observations (typically $\{\eta_t\}$)
- Some key features of the Lake Huron time series:
  - decreasing trend
  - some "random" fluctuations around the decreasing trend
- For example, we can extract the 'noise' component by assuming a linear trend



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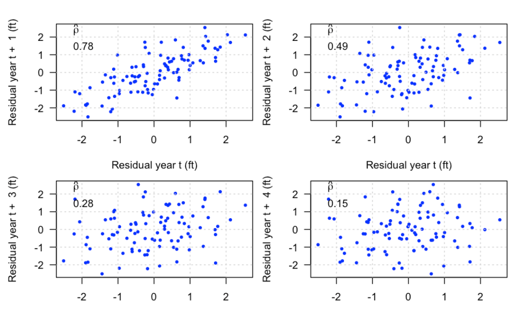
Exploring the Dependence Structure of "Noise" $\{\eta_t\}$

$\{\eta_t\}$ exhibit a temporal dependence structure, meaning that the nearby (in time) values tend to be more alike than those that are far part. To observe this, let's create a few time lag plots



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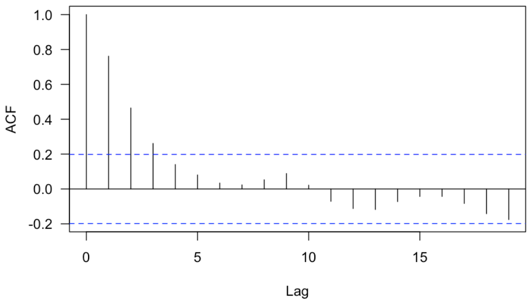
Further Exploration of the Temporal Dependence Structure

Let's plot the correlation as a function of the time lag



> We will learn how to use this information to suggest an appropriate time series model

Notes

Time Series Models

- A time series model is a probabilistic model that describes how the series data y_t could have been generated. More specifically, it is a probability model for $\{Y_t : t \in T\}$, a collection of random variables indexed in time
- We will keep our models for Y_t as simple as possible by assuming stationarity, meaning that some characteristics of the distribution of Y_t depend only on the "time lag" not on the specific time points
- While most time series are not stationary, we can model the non-stationary parts (e.g., by **de-trending** or **de-seasonalizing**) to obtain a stationary component, η_t. We typically assume the process is second-order stationary, meaning

$$\mathbb{E}[\eta_t] = 0, \quad \forall t \in T \quad \text{and,}$$
$$\mathrm{Cov}(\eta_t, \eta_{t'}) = \gamma(t' - t) = \mathrm{Cov}(\eta_{t+s}, \eta_{t'+s})$$

Notes

Time Series Models

- A time series model is a specification of the probabilistic distribution of a sequence of random variables (RVs) η_t

  (The observed time series is a realization of such a sequence of random variables)
- The simplest time series is i.i.d. (*independent and identically distributed*) noise
  - $\{\eta_t\}$ is a sequence of independent and identically distributed zero-mean (i.e., $\mathbb{E}(\eta_t) = 0, \forall t$) random variables ⇒ no temporal dependence
  - It is of little value of using i.i.d. noise model to conduct forecast as there is no information from the past observations
  - **But**, we will use i.i.d. model as a building block to develop time series models that can accommodate time dependence

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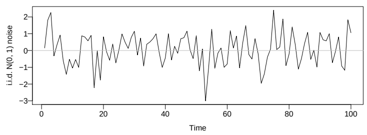
Example Realizations of i.i.d. Noise

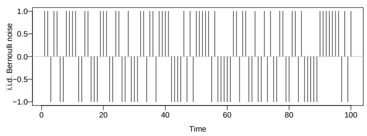
- Gaussian (normal) i.i.d. noise with mean 0 and variance $\sigma^2 > 0$

$$f(\eta_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\eta_t^2}{2\sigma^2})$$



- Bernoulli i.i.d. noise with "success" probability

$$\mathbb{P}(\eta_t = 1) = p = 1 - \mathbb{P}(\eta_t = -1)$$



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Means and Autocovarainces

A time series model could also be a specification of the means and autocovariances of the RVs

- The mean function of $\{\eta_t\}$ is

$$\mu_t = \mathbb{E}(\eta_t).$$

- μ_t is the population mean at time t, which can be computed as:

$$\mu_t = \begin{cases} \int_{-\infty}^{\infty} \eta_t f(\eta_t)\, d\eta_t & \text{when } \eta_t \text{ is a continuous RV;} \\ \sum_{-\infty}^{\infty} \eta_t p(\eta_t), & \text{when } \eta_t \text{ is a discrete RV,} \end{cases}$$

where $f(\cdot)$ and $p(\cdot)$ are the probability density function and probability mass function of η_t, respectively

Notes

Examples of Mean Functions

- **Example 1**: What is the mean function for $\{\eta_t\}$, an i.i.d. $\mathrm{N}(0, \sigma^2)$ process?

- **Example 2**: For each time point, let $Y_t = \beta_0 + \beta_1 t + \eta_t$ with β_0 and β_1 some constants and η_t is defined above. What is $\mu_Y(t)$?

Notes

Review: The Covariance Between Two RVs

- The covariance between the RVs X and Y is

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}\{(X-\mu_X)(Y-\mu_Y)\}\\ &= \mathbb{E}(XY) - \mu_X\mu_Y.\end{aligned}$$

  It is a measure of linear dependence between the two RVs. When $X = Y$ we have

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X).$$

- For constants a, b, c, and RVs X, Y, Z:

$$\begin{aligned}\mathrm{Cov}(aX+bY+c,Z) &= \mathrm{Cov}(aX,Z) + \mathrm{Cov}(bY,Z)\\ &= a\mathrm{Cov}(X,Z) + b\mathrm{Cov}(Y,Z)\end{aligned}$$

  $\Rightarrow$

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)\\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)\end{aligned}$$

Notes

Autocovariance Function

- The autocovariance function of $\{\eta_t\}$ is

$$\gamma(s,t) = \mathrm{Cov}(\eta_s,\eta_t) = \mathbb{E}[(\eta_s-\mu_s)(\eta_t-\mu_t)]$$

  It measures the strength of linear dependence between two RVs η_s and η_t

- **Properties**:
  - $\gamma(s,t) = \gamma(t,s)$ for each s and t
  - When $s = t$ we have

$$\gamma(t,t) = \mathrm{Cov}(\eta_t,\eta_t) = \mathrm{Cov}(\eta_t) = \sigma_t^2$$

    the value of the variance function at time t
  - $\gamma(s,t)$ is a non-negative definite function (will come back to this later)

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Autocorrelation Function

- The autocorrelation function of $\{\eta_t\}$ is

$$\rho(s,t) = \mathrm{Corr}(\eta_s,\eta_t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

  It measures the "scale invariant" linear association between η_s and η_t

- **Properties**:
  - $-1 \leq \rho(s,t) \leq 1$ for each s and t
  - $\rho(s,t) = \rho(t,s)$ for each s and t
  - $\rho(t,t) = 1$ for each t
  - $\rho(\cdot,\cdot)$ is a non-negative definite function

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Stationarity

- We typically need "replicates" to estimate population quantities. For example, we use
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
to be the estimate of μ_X, the population mean of the **single** RV, X
- However, in time series analysis, we have $n = 1$ (i.e., no replication) because we only have one realized value at each time point
- Stationarity means that some characteristic of $\{\eta_t\}$ does not depend on the time point, t, only on the "time lag" between time points **so that we can create "replicates"**

> Next, we will discuss strict stationarity and weak stationarity

Notes

Strictly Stationary Processes

- A time series, $\{\eta_t\}$, is strictly stationary if
$$[\eta_1, \eta_2, \cdots \eta_T] \overset{d}{=} [\eta_{1+h}, \eta_{2+h}, \cdots \eta_{T+h}],$$
for all integers h and $T \geq 1 \Rightarrow$ the joint distribution are unaffected by time shifts
- Under such the strict stationarity
  - $\{\eta_t\}$ is identically distributed but not (necessarily) independent
  - $\mu_t = \mu$ is independent of time t
  - $\gamma(s, t) = \gamma(s + h, t + h)$, for any s, t, and h

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Weakly Stationary Processes

- $\{\eta_t\}$ is weakly stationary if
  - $\mathbb{E}(\eta_t) = \mu_t = \mu$
  - $\text{Cov}(\eta_t, \eta_{t+h}) = \gamma(t, t + h) = \gamma(h)$, finite constant that can depend on h but not on t
- Other names for this type of stationarity include second-order, covariance, wide senese. The quantity h is called the lag
- Weak and strict stationarity
  - A strictly stationary process $\{\eta_t\}$ is also weakly stationary as long as μ is finite
  - Weak stationarity does not imply strict stationarity!

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Autocovariance Function of Stationary Processes

The autocovariance function (ACVF) of a stationary process $\{\eta_t\}$ is defined to be

$$\gamma(h) = \mathrm{Cov}(\eta_t, \eta_{t+h}) \\ = \mathbb{E}[(\eta_t - \mu)(\eta_{t+h} - \mu)],$$

which measures the lag-h time dependence

Properties of the ACVF:

- $\gamma(0) = \mathrm{Var}(\eta_t)$
- $\gamma(-h) = \gamma(h)$ for each h
- $\gamma(s-t)$ as a function of $(s-t)$ is non-negative definite

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Autocorrelation Function of Stationary Processes

The autocorrelation function (ACF) of a stationary process $\{\eta_t\}$ is defined to be

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

which measures the "scale invariant" lag-h time dependence

Properties of the ACF:

- $-1 \le \rho(h) \le 1$ and $\rho(0) = 1$ for each h
- $\rho(-h) = \rho(h)$ for each h
- $\rho(s-t)$ as a function of $(s-t)$ is non-negative definite

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Summary

In this lecture, we discuss

- Objectives of time series analysis
- Time series models
- Mean and auto-covariance/correlation functions
- Stationarity assumption in time series

The most important `R` function of this lecture is `acf`, which calculates and plots the sample autocorrelation

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