Lecture 13

Seasonal Time Series Models

Reading: Bowerman, O'Connell, and Koehler (2005): Chapter 11; Cryer and Chen (2008): Chapter 10

MATH 4070: Regression and Time-Series Analysis

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Agenda

1. Seasonal ARIMA (SARIMA) Model

2. A Case Study of Airline Passengers

Modeling Trend, Seasonality, and Noise

Recall the trend, seasonality, noise decomposition:

$$Y_t = \mu_t + s_t + \eta_t,$$

where

- μ_t: (deterministic) trend component;
- s_t: (deterministic) seasonal component with mean 0;
- η_t: random noise with $\mathbb{E}(\eta_t) = 0$

We have already described ways to estimate each component both separately and jointly (via likelihood-based method). But what about if $\{s_t\}$ is a "random" function of t?

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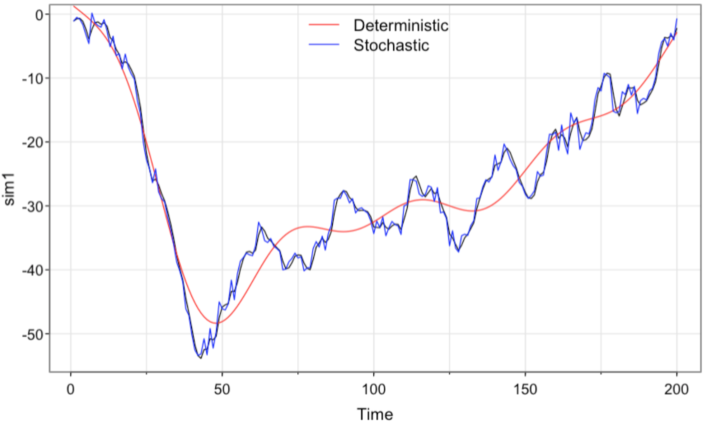
⇒ The seasonal ARIMA model allows us to model the case when s_t itself varies randomly from one cycle to the next

Digression: Using ARIMA for Stochastic Trend Modeling



For a given time series, it may be challenging to identify the exact form of a deterministic trend μ_t. However, ARIMA models can effectively capture and account for a "stochastic" trend

The Seasonal ARIMA (SARIMA) Model

Let d and D be non-negative integers. Then $\{X_t\}$ is a seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ process with period s if

$$Y_t = \nabla^d \nabla_s^D X_t = (1 - B)^d (1 - B^s)^D X_t,$$

is a casual ARMA process define by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

$\{Y_t\}$ is causal if $\phi(z) \neq 0$ and $\Phi(z) \neq 0$, for $|z| \leq 1$, where

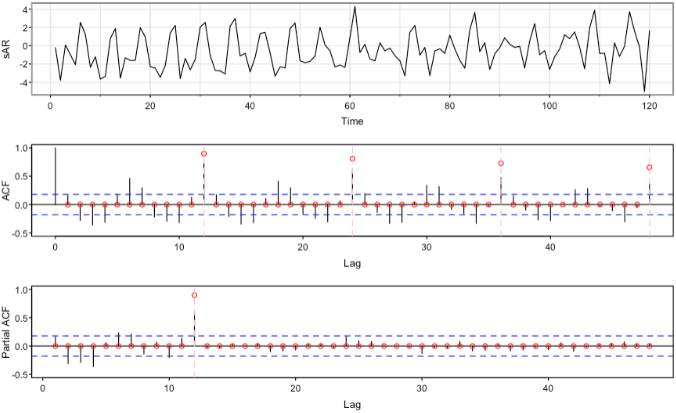
$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p;$$
$$\Phi(z) = 1 - \Phi_1 z - \cdots - \Phi_P z^P.$$

All roots of the AR and SAR characteristic equations must be greater than 1 in modulus

An Example of a Seasonal AR Model

$$Y_t = 0.9Y_{t-12} + Z_t,$$

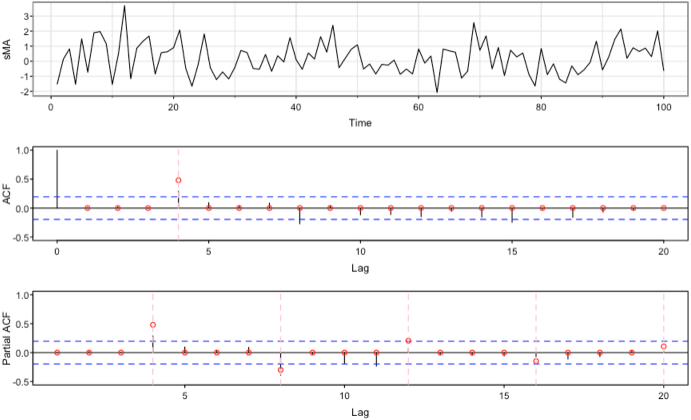
$$\Rightarrow p = q = d = D = Q = 0,\ P = 1,\ \Phi_1 = 0.9, s = 12.$$



An Example of a Seasonal MA Model

$$Y_t = Z_t + 0.75 Z_{t-4},$$

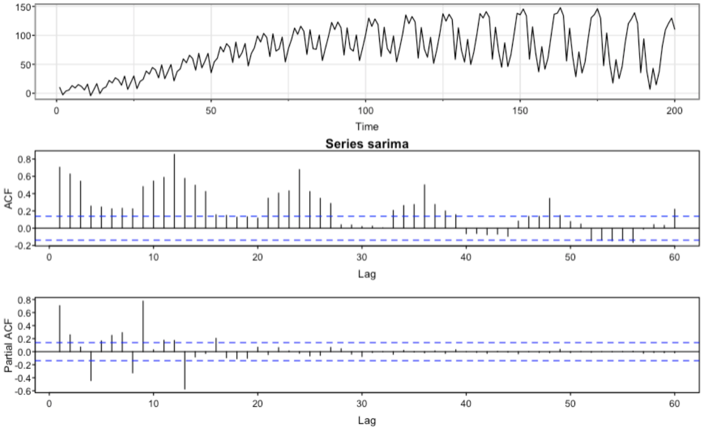
$$\Rightarrow p = q = d = D = P = 0,\ Q = 1,\ \Theta_1 = 0.75, s = 4.$$



Example of a SARIMA Model

$$(1-B)(1-B^{12})X_t = Y_t$$
$$(1+0.25B)(1-0.9B^{12})Y_t = (1+0.75B^{12})Z_t$$

$$\Rightarrow p = P = Q = d = D = 1,\ \phi = -0.25, \Phi = 0.9, \Theta_1 = 0.75, s = 12.$$



An Illustration of Seasonal Model

Consider a monthly time series $\{X_t\}$ with both a trend, and a seasonal component of period $s = 12$.

- Suppose we know the values of d and D such that $Y_t = (1 - B)^d (1 - B^{12})^D X_t$ is stationary
- We can arrange the data this way:

	Month 1	Month 2	$\cdots$	Month 12
Year 1	Y_1	Y_2	$\cdots$	Y_{12}
Year 2	Y_{13}	Y_{14}	$\cdots$	Y_{24}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
Year r	$Y_{1+12(r-1)}$	$Y_{2+12(r-1)}$	$\cdots$	$Y_{12+12(r-1)}$

The Inter-annual Model

Here we view each column (month) of the data table from the previous slide as a separate time series

- For each month m, we assume the same ARMA(P, Q) model. We have

$$Y_{m+12y} - \sum_{i=1}^{P} \Phi_i Y_{m+12(y-i)} = U_{m+12y} + \sum_{j=1}^{Q} \Theta_j U_{m+12(y-j)},$$

for each $y = 0, \cdots, r-1$, where $\{U_{m+12y:y=0,\cdots,r-1}\} \sim \mathrm{WN}(0, \sigma_U^2)$ for each m

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- We can write this as

$$\Phi(B^{12}) Y_t = \Theta(B^{12}) U_t,$$

  and this defines the inter-annual model

The Intra-Annual Model

We induce correlation between the months by letting the process $\{U_t\}$ follow an ARMA(p, q) model,

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- This is the intra-annual model
- The combination of the inter-annual and intra-annual models for the differenced stationary series,

$$Y_t = (1 - B)^d (1 - B^{12})^D X_t,$$

  yields a SARIMA model for $\{X_t\}$

Steps for Modeling SARIMA Processes

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4. Examine the sample ACF/PACF at lags $\{1, 2, \cdots, s - 1\}$, to identify possible values of p and q

Modeling SARIMA Processes (Cont'd)

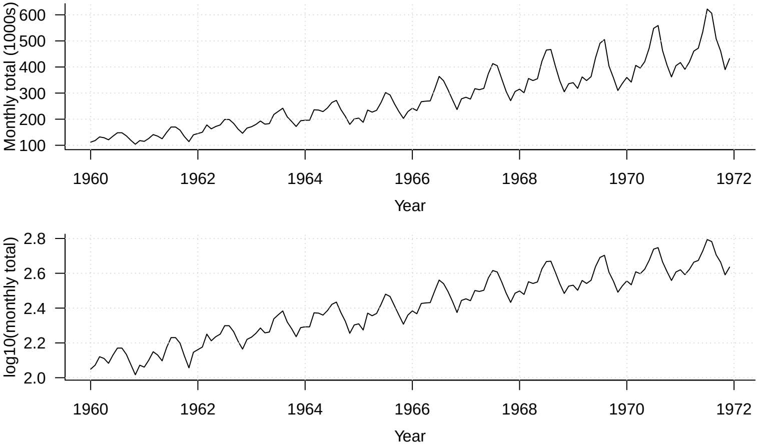
5. Use maximum likelihood method to fit the models

6. Use model summaries, diagnostics, AIC (AICc) to determine the best SARIMA model

7. Conduct forecast

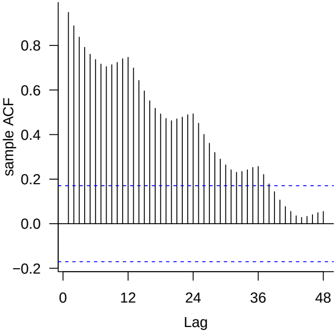
Airline Passengers Example

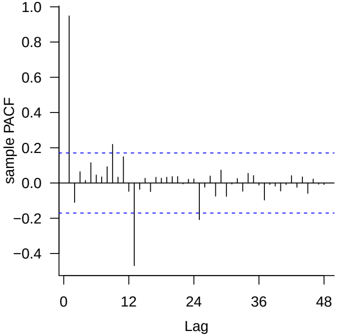
We consider the data set `airpassengers`, which are the monthly totals of international airline passengers from 1960 to 1971.



Here we stabilize the variance with a $\log_{10}$ transformation

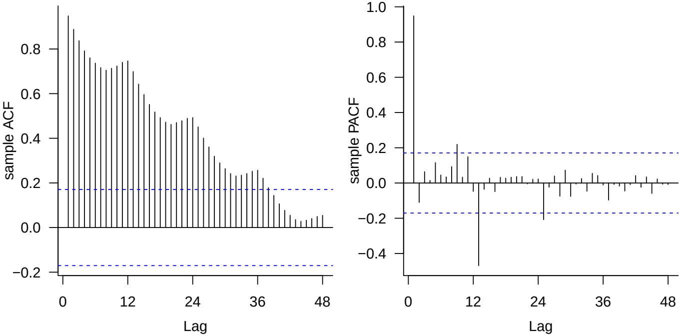
Sample ACF/PACF Plots





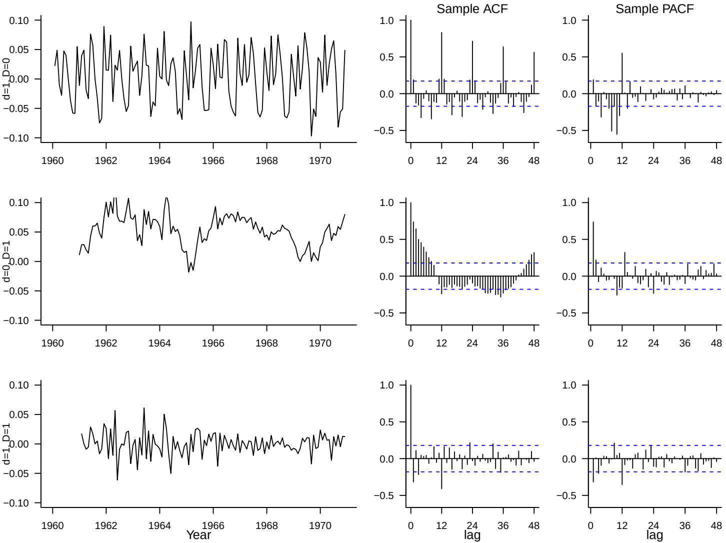
- The sample ACF decays slowly with a wave structure ⇒ seasonality

Sample ACF/PACF Plots



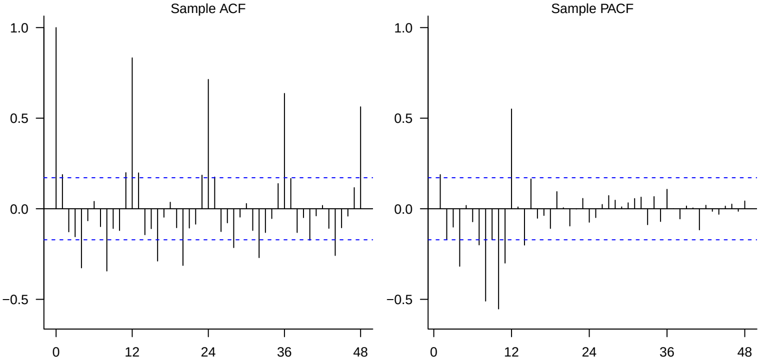
- The sample ACF decays slowly with a wave structure ⇒ seasonality
- The lag one PACF is close to one, indicating that differencing the data would be reasonable

Trying Different Orders of Differencing



Choosing Candidate SARIMA Models

We choose a SARIMA$(p, 1, q) \times (P, 0, Q)_{12}$ model. Next we examine the sample ACF/PACF of the process $Y_t = (1 - B)X_t$

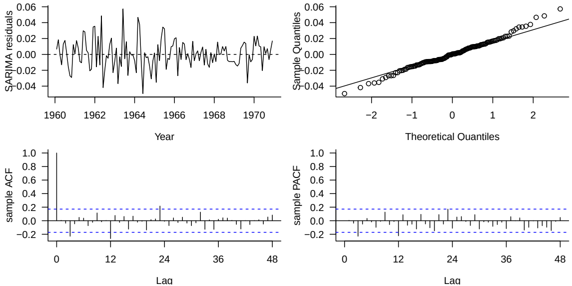


Now we need to choose P, Q, p, and q

Fitting a SARIMA$(1,1,0)\times(1,0,0)$ Model

```
> fit1 <- arima(diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0), period = 12))
> fit1

Call:
arima(x = diff.1.0, order = c(1, 0, 0), seasonal = list(order = c(1, 0, 0),
    period = 12))

Coefficients:
          ar1    sar1  intercept
      -0.2667  0.9291     0.0039
s.e.   0.0865  0.0235     0.0096

sigma^2 estimated as 0.0003298:  log likelihood = 327.27,  aic = -646.54
```

```
> Box.test(fit1$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit1$residuals
X-squared = 55.372, df = 48, p-value = 0.2164
```



A Discussion of the Model Fit

- Residuals show greater spread in 1949-1955 and have heavier-than-normal tails
- The Ljung-Box test result indicates the fitted SARIMA $(1,1,0) \times (1,0,0)_{12}$ has sufficiently account for the temporal dependence
- 95% CI for ϕ_1 and Φ_1 do not contain zero $\Rightarrow$ no need to go with simpler model

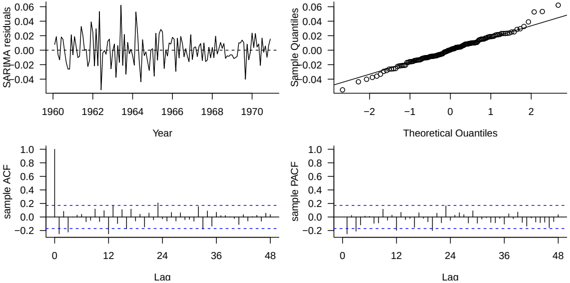
Our estimated model is:

$$X_t = \log_{10}(\text{\#Passengers})$$
$$Y_t = (1-B)X_t = X_t - X_{t-1}$$
$$(1+0.2667B)(1-0.9291B^{12})(Y_t - 0.0039) = Z_t,$$

where $\{Z_t\} \overset{i.i.d.}{\sim} \mathrm{N}(0,\sigma^2)$ with $\hat{\sigma}^2 = 0.00033$

Comparing with a SARIMA$(0,1,0) \times (1,0,0)$ Model

```
> (fit2 <- arima(diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12)))

Call:
arima(x = diff.1.0, seasonal = list(order = c(1, 0, 0), period = 12))

Coefficients:
        sar1  intercept
      0.9081     0.0040
s.e.  0.0278     0.0108

sigma^2 estimated as 0.0003616:  log likelihood = 322.75,  aic = -639.51
> Box.test(fit2$residuals, lag = 48, type = "Ljung-Box")

        Box-Ljung test

data:  fit2$residuals
X-squared = 80.641, df = 48, p-value = 0.002209
```



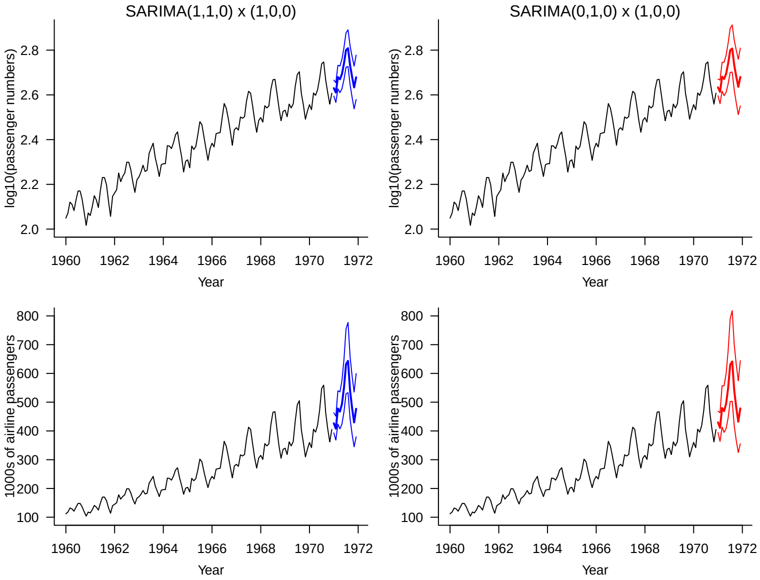
A Discussion of SARIMA$(0,1,0) \times (1,0,0)$ Model Fit

Here we drop the AR(1) term

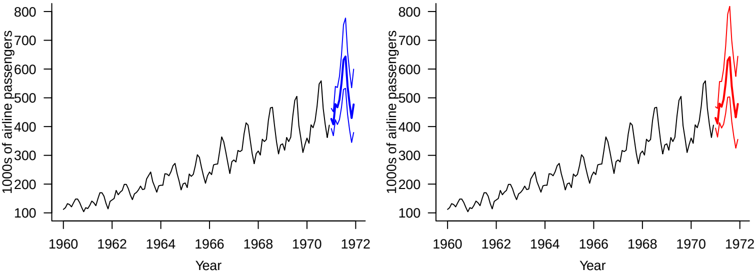
- Residual plots are similar to before, with greater spread in 1949-1955 and heavy tails
- Both $\hat{\sigma}^2$ and AIC increase (compared with model fit1)
- The lag 1 of ACF and PACF now lies outside the IID noise bounds. The Ljung-Box p-value of 0.0022, leads us to reject the IID residual assumption

In conclusion, the SARIMA$(1,1,0) \times (1,0,0)_{12}$ model fits better than SARIMA$(0,1,0) \times (1,0,0)_{12}$

Forecasting the 1971 Data



Evaluating Forecast Performance



Metrics	Model Fit1	Model Fit2
Root Mean Square Error	30.36	31.32
Mean Relative Error	0.057	0.060
Empirical Coverage	0.917	1.000

The SARIMA$(1,1,0)\times(1,0,0)$ Model is Equivalent To?

Our model for the log passenger series $\{X_t\}$ is

$$\phi(B)\Phi(B^{12})(1-B)X_t = Z_t,$$

where $\phi(B) = 1 - \phi_1 B$ and $\Phi(B) = 1 - \Phi_1(B)$

Note that

$$\begin{aligned}\phi(B)\Phi(B^{12}) &= (1-\phi_1 B)(1-\Phi_1 B^{12}) \\ &= 1 - \phi_1 B - \Phi_1 B^{12} + \phi_1 \Phi_1 B^{13}\end{aligned}$$

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Question: What is this SARIMA model equivalent to?