School of MATHEMATICAL AND STATISTICAL SCIENCES
Clemson University

Review
Some Examples of Stationary Processes
Estimation and Inference for Mean Functions
Differencing

Lecture 7

Stationary Processes

Reading: Bowerman, O'Connell, and Koehler (2005): Chapter; Cryer and Chen (2008): Chapter 1.2; Chapter 4.2 and 4.3

MATH 4070: Regression and Time-Series Analysis

Whitney Huang
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Agenda

1. **Review**
2. **Some Examples of Stationary Processes**
3. **Estimation and Inference for Mean Functions**
4. **Differencing**

Time Series Modeling Strategy

Additive Decompstion:

$$Y_t = \mu_t + s_t + \eta_t, \quad t = 1, 2, \cdots, T$$

1. Plot the data y_t to explore the form of μ_t and s_t, and check for non-constant variation in η_t

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6. Check for fit of model (poor fit $\Rightarrow$ return to step 1)
7. Use model for inference: predicting future y_t's, describing changes in y_t over time, hypothesis testing, etc

Recap of the Past Few Lectures

- We discussed the use of regression techniques to model the (deterministic) μ_t and s_t

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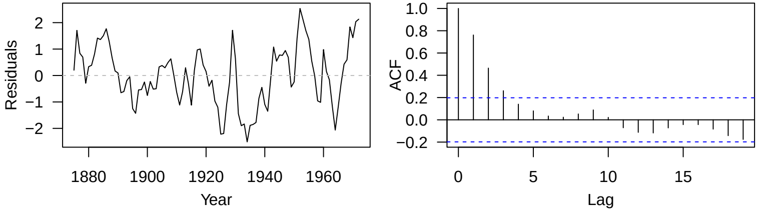
- Residuals typically suggest temporal dependence in $\{\eta_t\}$
- Time series models concern the modeling of temporal dependence in $\{\eta_t\}$
- Stationarity assumption typically employed to overcome the issue of "one sample"
- Weakly stationary: constant mean and variance over time, with covariance depending only on time lags

The Implications of Temporal Dependence



- There is a consistent relationship between conservative residuals
- The usual regression assumptions are violated, and t- and F-tests are not valid ☹
- We can get better predictions of future values by modeling autocorrelation ☺

The White Noise Process

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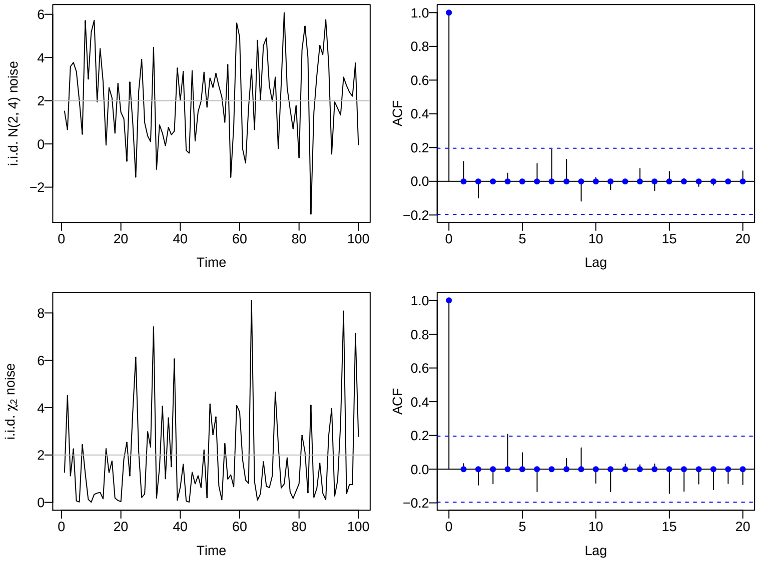
Let's assume $\mathbb{E}(\eta_t) = \mu$ and $\mathrm{Var}(\eta_t) = \sigma^2 < \infty$. $\{\eta_t\}$ is a white noise or $\mathrm{WN}(\mu, \sigma^2)$ process if

$$\gamma(h) = 0,$$

for $h \neq 0$

- $\{\eta_t\}$ is stationary
- However, distributions of η_t and η_{t+1} can be different!
- All i.i.d. noise with finite variance ($\sigma^2 < 0$) is white noise but the converse need not be true

Examples Realizations of White Noise Processes



The Moving Average Process of First Order: MA(1)

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Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and θ be some constant $\in \mathbb{R}$. For each integer t, let

$$\eta_t = Z_t + \theta Z_{t-1}.$$

- The sequences of RVs $\{\eta_t\}$ is called the moving average process of order 1 or MA(1) process
- One can show that the MA(1) process $\{\eta_t\}$ is stationary

MA(1): Mean Function

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Need to show the mean function is NOT a function of time t

$$\begin{aligned}\mathbb{E}[\eta_t] &= \mathbb{E}[Z_t + \theta Z_{t-1}] \\ &= \mathbb{E}[Z_t] + \theta \mathbb{E}[Z_{t-1}] \\ &= 0 + \theta \times 0 \\ &= 0, \quad \forall t\end{aligned}$$

☺

MA(1): Covariance Function

Need to show the autovariance function $\gamma(\cdot,\cdot)$ is a function of time lag only

$$\begin{aligned}
\gamma(t,t+h) &= \mathrm{Cov}(\eta_t,\eta_{t+h}) \\
&= \mathrm{Cov}(Z_t+\theta Z_{t-1}, Z_{t+h}+\theta Z_{t+h-1}) \\
&= \mathrm{Cov}(Z_t, Z_{t+h}) + \mathrm{Cov}(Z_t, \theta Z_{t+h-1}) \\
&+ \mathrm{Cov}(\theta Z_{t-1}, Z_{t+h}) + \mathrm{Cov}(\theta Z_{t-1}, \theta Z_{t+h-1})
\end{aligned}$$

MA(1): Covariance Function

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if $h=0,$ we have $\quad \gamma(t,t+h) = \sigma^2 + \theta^2\sigma^2 = \sigma^2(1+\theta^2)$
if $h=\pm 1,$ we have $\quad \gamma(t,t+h) = \theta\sigma^2$
if $|h| \geq 2,$ we have $\quad \gamma(t,t+h) = 0$

$\Rightarrow \gamma(t,t+h)$ only depends on h but not on t ☺

MA(1): ACVF & ACF

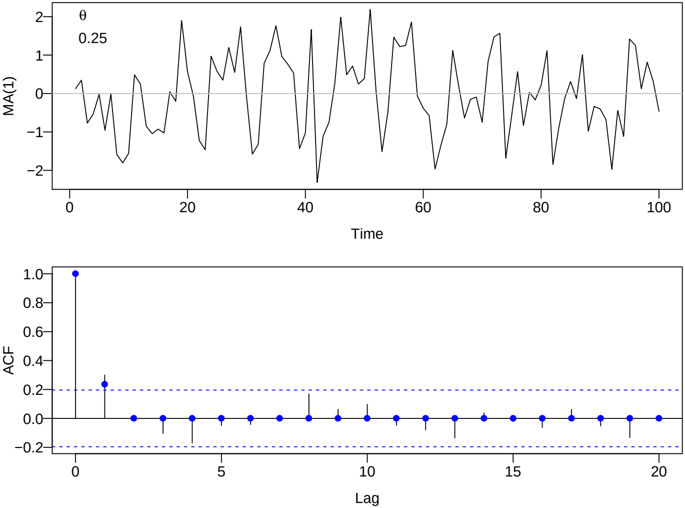
ACVF:

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0; \\ \theta\sigma^2 & |h| = 1; \\ 0 & |h| \geq 2 \end{cases}$$

We can get **ACF** by dividing everything by $\gamma(0) = \sigma^2(1+\theta^2)$

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \frac{\theta}{1+\theta^2} & |h| = 1; \\ 0 & |h| \geq 2. \end{cases}$$

Examples Realizations of MA(1) Processes



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First-order autoregressive process: AR(1)

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process, and $-1 < \phi < 1$ be a constant. Let's assume $\{\eta_t\}$ is a stationary process with

$$\eta_t = \phi \eta_{t-1} + Z_t,$$

for each integer t, where η_s and Z_t are uncorrelated for each $s < t \Rightarrow$ future noise is uncorrelated with the current time point

We will see later there is only one unique solution to this equation. Such a sequence $\{\eta_t\}$ of RVs is called an AR(1) process

Properties of the AR(1) process

Want to find the mean value μ under the weakly stationarity assumption

$$\begin{aligned}
\mathbb{E}[\eta_t] &= \mathbb{E}[\phi\eta_{t-1} + Z_t] \\
\mu &= \phi\mathbb{E}[\eta_{t-1}] + \mathbb{E}[Z_t] \\
\mu &= \phi\mu + 0 \\
&\Rightarrow \mu = 0, \quad \forall t
\end{aligned}$$

☺

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☺

Want to find $\gamma(h)$ under the weakly stationarity assumption

$$\begin{aligned}
\mathrm{Cov}(\eta_t, \eta_{t-h}) &= \mathrm{Cov}(\phi\eta_{t-1} + Z_t, \eta_{t-h}) \\
\gamma(-h) &= \phi\mathrm{Cov}(\eta_{t-1}, \eta_{t-h}) + \mathrm{Cov}(Z_t, \eta_{t-h}) \\
\gamma(h) &= \phi\gamma(h-1) + 0 \\
\Rightarrow \gamma(h) &= \phi\gamma(h-1) = \cdots = \phi^{|h|}\gamma(0)
\end{aligned}$$

Next, need to figure out $\gamma(0)$

Properties of the AR(1) process Cont'd

$$\begin{aligned}
\mathrm{Var}(\eta_t) &= \mathrm{Var}(\phi\eta_{t-1} + Z_t) \\
\gamma(0) &= \phi^2\gamma(0) + \sigma^2 \\
\Rightarrow (1-\phi^2)\gamma(0) &= \sigma^2 \\
\Rightarrow \gamma(0) &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

☺

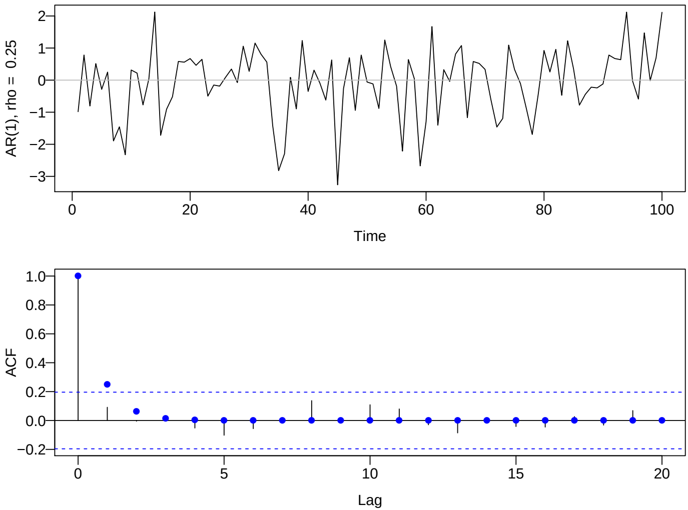
Therefore, we have

$$\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & h = 0; \\ \frac{\phi^{|h|}\sigma^2}{1-\phi^2} & |h| \geq 1, \end{cases}$$

and

$$\rho(h) = \begin{cases} 1 & h = 0; \\ \phi^{|h|} & |h| \geq 1. \end{cases}$$

Examples Realizations of AR(1) Processes



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The Random Walk Process

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and for $t \geq 1$ definite

$$\eta_t = Z_1 + Z_2 + \cdots + Z_t = \sum_{s=1}^{t} Z_s.$$

- The sequence of RVs $\{\eta_t\}$ is called a random walk process
- **Special case**: If we have $\{Z_t\}$ such that for each t

$$\mathbb{P}(Z_t = z) = \begin{cases} \frac{1}{2}, & z = 1; \\ \frac{1}{2}, & z = -1, \end{cases}$$

  then $\{\eta_t\}$ is a simple symmetric random walk
- The random walk process is not stationary!

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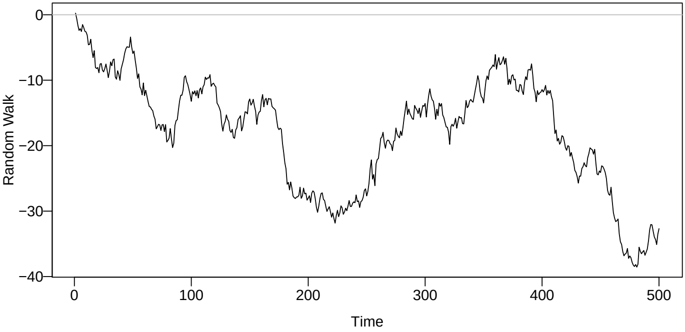
Example Realizations of Random Walk Processes



Gaussian Processes

$\{\eta_t\}$ is a Gaussian process (GP) if the joint distribution of any collection of the RVs has a multivariate normal (aka Gaussian) distribution

- The distribution of a GP is fully characterized by $\mu(\cdot)$, the mean function, and $\gamma(\cdot,\cdot)$, the autocovariance function. The joint probability density function of $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_T)^T$ is

$$f(\boldsymbol{\eta}) = \frac{1}{(2\pi)^{\frac{T}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\mu})^T \Sigma^{-1}(\boldsymbol{\eta}-\boldsymbol{\mu})\right),$$

  where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_T)^T$ and the (i,j) element of the covariance matrix Σ is $\gamma(i,j)$

- If a GP $\{\eta_t\}$ is weakly stationary then the process is also strictly stationary

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Estimating the Mean of Stationary Processes

Let $\{\eta_t\}$ be stationary with mean μ and ACVF $\gamma(s,t) = \gamma(s-t)$

- A natural estimator of μ is the sample mean

$$\bar{\eta} = \frac{1}{T}\sum_{t=1}^{T}\eta_t.$$

$\bar{\eta}$ is an unbiased estimator of μ, i.e.

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- Since $\{\eta_t\}$ is stationary, we have

$$\begin{aligned}\mathrm{Var}(\bar{\eta}) &= \frac{1}{T^2}\mathrm{Var}\left(\sum_{i=1}^{T}\eta_t\right)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{Cov}(\eta_s,\eta_t)\\ &= \frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\gamma(s-t)\end{aligned}$$

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- **Exercise**: Show

$$\mathrm{Var}(\bar{\eta}) = \frac{1}{T}\sum_{h=-(T-1)}^{T-1}\left(1-\frac{|h|}{T}\right)\gamma(h)$$

AR(1) Example

Suppose $\{\eta_1, \eta_2, \eta_3\}$ is an AR(1) process with $|\phi| < 1$ and innovation variance σ^2. Show that the variance of $\bar{\eta}$ is $\frac{\sigma^2}{9(1-\phi^2)}(3 + 4\phi + 2\phi^2)$

Solution:

The Sampling Distribution of $\bar{\eta}$

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Let

$$v_T = \sum_{h=-(T-1)}^{(T-1)} \left(1 - \frac{|h|}{T}\right) \gamma(h)$$

- If $\{\eta_t\}$ is Gaussian we have

$$\sqrt{T}(\bar{\eta} - \mu) \sim \mathrm{N}(0, v_T)$$

- The result above is approximate for many non-Gaussian time series
- In practice we also need to estimate $\gamma(h)$ from the data

Confidence Intervals for μ

- If $\gamma(h) \to 0$ as $h \to \infty$ then

$$v = \lim_{T\to\infty} v_T = \sum_{h=-\infty}^{\infty} \gamma(h) \text{ exists.}$$

- Further, if $\{\eta_t\}$ is Gaussian and

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then an approximate large-sample 95% CI for μ is given by

$$\left[\bar{\eta} - 1.96\sqrt{\frac{v}{T}}, \bar{\eta} + 1.96\sqrt{\frac{v}{T}}\right]$$

Strategies for Estimating v

- Parametric:
  - Assume a parametric model $\gamma_{\boldsymbol{\theta}}(\cdot)$, and calculate

    $$\hat{v} = \sum_{h=-\infty}^{\infty} \gamma_{\hat{\boldsymbol{\theta}}}(h)$$

    based on the ACVF for that model
  - The standard error, v, will depend on the parameters $\boldsymbol{\theta}$ of the parametric model
- Nonparametric:
  - Estimate v by

    $$\hat{v} = \sum_{h=-\infty}^{\infty} \hat{\gamma}(h),$$

    where $\hat{\gamma}(\cdot)$ is an nonparametric estimate of ACVF

Examples of Parametric Forms for v

- i.i.d. Gaussian Noise: $v = \gamma(0) = \sigma^2 \Rightarrow$ CI reduces to the classical case:

$$\left[\bar{\eta} - 1.96\sqrt{\frac{\sigma^2}{T}}, \bar{\eta} + 1.96\sqrt{\frac{\sigma^2}{T}}\right]$$

- MA(1) process: We have

$$\begin{aligned} v = \sum_{h=-\infty}^{\infty} \gamma(h) &= \gamma(-1) + \gamma(0) + \gamma(1) \\ &= \gamma(0) + 2\gamma(1) \\ &= \sigma^2(1 + \theta^2 + 2\theta) = \sigma^2(1 + \theta)^2 \end{aligned}$$

- **Exercise**: Show for an AR(1) process we have

$$v = \frac{\sigma^2}{(1 - \phi)^2}$$

Differencing

Instead of modeling trends, one can consider removing trends by differencing

- Define the first order difference operator ∇ as

$$\nabla Y_t = Y_t - Y_{t-1} = (1 - B)Y_t,$$

 where B is the **backshift operator** and is defined as $BY_t = Y_{t-1}$.

- Similarly the general order difference operator $\nabla^q Y_t$ is **defined recursively** as $\nabla[\nabla^{q-1} Y_t]$

- The backshift operator of power q is defined as $B^q Y_t = Y_{t-q}$

In next slide we will see an example regarding the relationship between ∇^q and B^q

Difference and Backshift Notation

The second order difference is given by

$$\nabla^2 Y_t = \nabla[\nabla Y_t]$$

Difference and Backshift Notation

The second order difference is given by

$$\begin{aligned}\nabla^2 Y_t &= \nabla[\nabla Y_t] \\ &= \nabla[Y_t - Y_{t-1}]\end{aligned}$$

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Difference and Backshift Notation

The second order difference is given by

$$
\begin{aligned}
\nabla^2 Y_t &= \nabla[\nabla Y_t] \\
&= \nabla[Y_t - Y_{t-1}] \\
&= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})
\end{aligned}
$$

Difference and Backshift Notation

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In the next slide we will see an example of using differening to remove the trend

Removing Trend via Differening

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Consider a time series data with a linear trend (i.e., $\{Y_t = \beta_0 + \beta_1 t + \eta_t\}$) where η_t is a stationary time series. Then first order differencing results in a stationary series with no trend. To see why

$$\begin{aligned}\nabla Y_t &= Y_t - Y_{t-1} \\ &= (\beta_0 + \beta_1 t + \eta_t) - (\beta_0 + \beta_1 (t-1) + \eta_{t-1}) \\ &= \beta_1 + \eta_t - \eta_{t-1}\end{aligned}$$

This is the sum of a stationary series and a constant, and therefore we have successfully remove the linear trend.

Notes on Differening

Review

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Differencing

- A polynomial trend of order q can be removed by q-th order differencing
- By q-th order differencing a time series we are shortening its length by q
- Differencing does not allow you to estimate the trend, only to remove it. *Therefore it is not appropriate if the aim of the analysis is to describe the trend*

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Seasonal Differening

- The lag-d difference operator, ∇_d, is defined by

$$\nabla_d Y_t = Y_t - Y_{t-d} = (1 - B^d)Y_t.$$

Note: This is NOT ∇^d!

- **Example**: Consider data that arise from the model $Y_t = \beta_0 + \beta_1 t + s_t + \eta_t$, which has a linear trend and seasonal component that repeats itself every d time points. Then by just seasonal differencing (lag-d differening here) this series becomes stationary.

$$\begin{aligned}\nabla_d Y_t &= Y_t - Y_{t-d} \\ &= [\beta_0 + \beta_1 t + s_t + \eta_t] - [\beta_0 + \beta_1 (t - d) + s_{t-d} + \eta_{t-d}] \\ &= d\beta_1 + \eta_t - \eta_{t-d}\end{aligned}$$

Summary

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In this lecture, we discuss

- White Noise Processes, MA(1), AR(1)
- Estimation and Inference of the Mean of Stationary Processes
- Differencing to Remove Trend and Seasonality

The most important `R` function for this lecture is `arima.sim`, which can be used to simulate MA(1), AR(1), and more general ARIMA models