Autocovariance Estimation and Testing

Linear Processes

Autoregressive-Moving Average Model: Stationarity, Causality, and Invertibility

Partial Autocorrelation Functions

Lecture 8

Autoregressive Moving-Average (ARMA) Models

Reading: Bowerman, O'Connell, and Koehler (2005): Chapter 9; Cryer and Chen (2008): Chapter 4

MATH 4070: Regression and Time-Series Analysis

Whitney Huang
Clemson University

Agenda

1. Autocovariance Estimation and Testing

2. Linear Processes

3. Autoregressive-Moving Average Model: Stationarity, Causality, and Invertibility

4. Partial Autocorrelation Functions

School of MATHEMATICAL AND STATISTICAL SCIENCES Clemson University

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Estimation of Autocovariance Function $\gamma(\cdot)$

Goal: Want to estimate

$$\gamma(h) = \mathrm{Cov}(\eta_t, \eta_{t+h}) = \mathbb{E}\left[(\eta_t - \mu)(\eta_{t+h} - \mu)\right]$$

using data $\{\eta_t\}_{t=1}^{T}$

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using data $\{\eta_t\}_{t=1}^{T}$

For $|h| < T$, consider

$$\hat{\gamma}(h) = \frac{1}{T} \sum_{t=1}^{T-|h|} (\eta_t - \bar{\eta})(\eta_{t+|h|} - \bar{\eta}).$$

We call $\hat{\gamma}(h)$ the sample ACVF

- The sample ACVF $\hat{\gamma}(h)$ is used as the **standard** estimate of $\gamma(h)$ and is even and non-negative definite

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- The sample ACVF $\hat{\gamma}(h)$ is used as the **standard** estimate of $\gamma(h)$ and is even and non-negative definite
- The sample ACVF is a biased estimator of $\gamma(h)$, that is, $\mathbb{E}[\hat{\gamma}(h)] \neq \gamma(h)$

The Sample Autocorrelation Function

- The sample autocorrelation function (ACF) is defined for $|h| < T$ by

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

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- **Rule of thumb**: Box and Jenkins (1976) recommend using $\hat{\rho}(h)$ and $\hat{\gamma}(h)$ only for $\frac{|h|}{T} \leq \frac{1}{4}$ and $T \geq 50$

The Sample Autocorrelation Function

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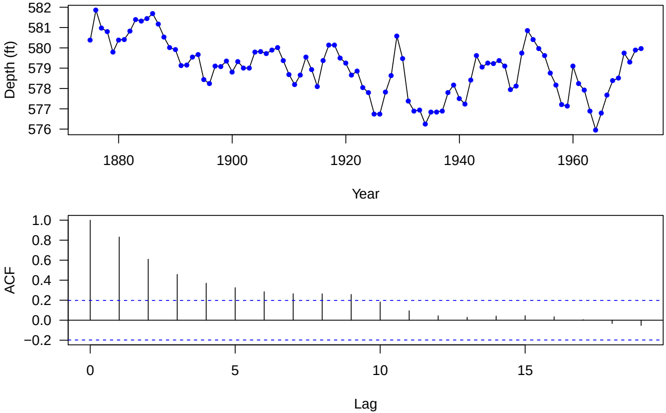
- **Rule of thumb**: Box and Jenkins (1976) recommend using $\hat{\rho}(h)$ and $\hat{\gamma}(h)$ only for $\frac{|h|}{T} \leq \frac{1}{4}$ and $T \geq 50$
- This is because estimates $\hat{\rho}(h)$ and $\hat{\gamma}(h)$ are unstable for large $|h|$ as there will be no enough data points going into the estimator

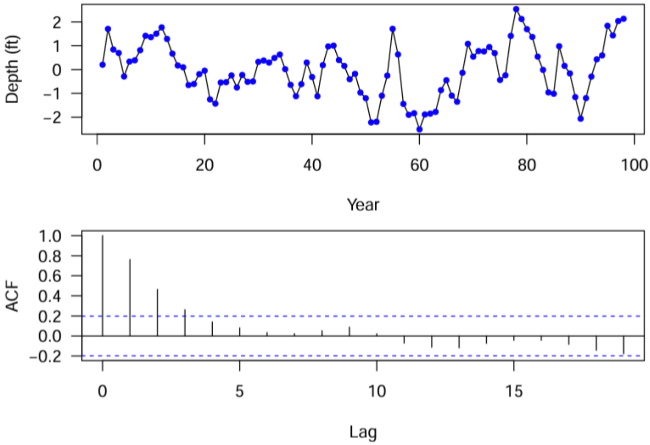
Calculating the Sample ACF in R

- Use `acf` function to calculate the sample ACF
- Lake Huron Example (`acf(LakeHuron)`–note that this is NOT the right thing to do here; see the next slide))



Sample ACF for the Lake Huron Example

- Recall that the ACF is used to characterize a stationary process
- Ensure the series is (approximately) stationary; if not, model and remove the non-stationary component.



Asymptotic Distribution of the Sample ACF [Bartlett, 1946]

Let $\{\eta_t\}$ be a stationary process we suppose that the ACF

$$\boldsymbol{\rho} = (\rho(1), \rho(2), \cdots, \rho(k))^T$$

is estimated by

$$\hat{\boldsymbol{\rho}} = (\hat{\rho}(1), \hat{\rho}(2), \cdots, \hat{\rho}(k))^T$$

- For large T

  $$\hat{\boldsymbol{\rho}} \dot{\sim} \mathrm{N}_k(\boldsymbol{\rho}, \frac{1}{T}W),$$

  where N_k is the k-variate normal distribution and W is an $k \times k$ covariance matrix with (i, j) element defined by

  $$w_{ij} = \sum_{h=1}^{\infty} a_{ih} a_{jh}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq k$$

  where $a_{ih} = \rho(h+i) + \rho(h-i) - 2\rho(h)\rho(i)$

Using the ACF as a Test for i.i.d. Noise

When $\{\eta_t\}$ is an **i.i.d. process** with finite variance, Bartlett's result simplifies for each $h \neq 0$

$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{T}).$$

This suggests a diagnostic for i.i.d. noise:

1. Plot the lag h versus the sample ACF $\hat{\rho}(h)$

Using the ACF as a Test for i.i.d. Noise

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This suggests a diagnostic for i.i.d. noise:

1. Plot the lag h versus the sample ACF $\hat{\rho}(h)$
2. Draw two horizontal lines at $\pm\frac{1.96}{\sqrt{T}}$ (blue dashed lines in R)

Using the ACF as a Test for i.i.d. Noise

When $\{\eta_t\}$ is an **i.i.d. process** with finite variance, Bartlett's result simplifies for each $h \neq 0$

$$\hat{\rho}(h) \dot{\sim} \mathrm{N}(0, \frac{1}{T}).$$

This suggests a diagnostic for i.i.d. noise:

1. Plot the lag h versus the sample ACF $\hat{\rho}(h)$
2. Draw two horizontal lines at $\pm \frac{1.96}{\sqrt{T}}$ (blue dashed lines in `R`)
3. About 95% of the $\{\hat{\rho}(h) : h = 1, 2, 3, \cdots\}$ should be within the lines **if we have i.i.d. noise**

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The Portmanteau Test [Box and Pierce, 1970] for i.i.d. Noise

Suppose we wish to test:

$H_0 : \{\eta_1, \eta_2, \cdots, \eta_T\}$ is an i.i.d. noise sequence

$H_1 : H_0$ is false

- Under H_0,

$$\hat{\rho}(h) \mathrel{\dot\sim} \mathrm{N}(0, \frac{1}{T}) \stackrel{d}{=} \frac{1}{\sqrt{T}}\mathrm{N}(0,1)$$

- Hence

$$Q = T\sum_{i=1}^{k} \hat{\rho}^2(h) \mathrel{\dot\sim} \chi^2_{df=k}$$

- We reject H_0 if $Q > \chi^2_k(1-\alpha)$, the $1-\alpha$ quatile of the chi-squared distribution with k degrees of freedom

Ljung-Box Test [Ljung and Box, 1978]

Ljung and Box [1978] showed that

$$Q_{\text{LB}} = T(T-2)\sum_{h=1}^{k}\frac{\hat{\rho}^2(h)}{T-h} \mathrel{\dot{\sim}} \chi^2_k.$$

The Ljung-Box test can be more powerful than the Portmanteau test

Both the Portmanteau Test (aka Box-Pierce test) and Ljung-Box test can be carried out in `R` using the function `Box.test`, with the options `type = c("Box-Pierce", "Ljung-Box")`

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Examples in R

```
> Box.test(rnorm(100), 20)

        Box-Pierce test

data:  rnorm(100)
X-squared = 12.197, df = 20, p-value = 0.9091

> Box.test(LakeHuron, 20)

        Box-Pierce test

data:  LakeHuron
X-squared = 182.43, df = 20, p-value < 2.2e-16

> Box.test(LakeHuron, 20, type = "Ljung")

        Box-Ljung test

data:  LakeHuron
X-squared = 192.6, df = 20, p-value < 2.2e-16
```

Linear Processes

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- A time series $\{\eta_t\}$ is a linear process with mean μ if we can write it as

$$\eta_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_j, \quad \forall t,$$

  where μ is a real-valued constant, $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process and $\{\psi_j\}$ is a set of absolutely summable constants[1]

- Absolute summability of the constants guarantees that the infinite sum converges

[1] A set of real-valued constants $\{\psi_j : j \in \mathbb{Z}\}$ is absolutely summable if $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Example: Moving Average Process of Order q, MA(q)

Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process. For an integer $q > 0$ and constants $\theta_1, \cdots, \theta_q$ with $\theta_q \neq 0$, define

$$\begin{aligned} \eta_t &= Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \\ &= \theta_0 Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} \\ &= \sum_{j=0}^{q} \theta_j Z_{t-j}, \end{aligned}$$

where we let $\theta_0 = 1$

$\{\eta_t\}$ is known as the moving average process of order q, or the MA(q) process, and, by definition, is a linear process

Defining Linear Processes with Backward Shifts

- Recall the backward shift operator, B, is defined by $B\eta_t = \eta_{t-1}$

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Defining Linear Processes with Backward Shifts

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- Recall the backward shift operator, B, is defined by $B\eta_t = \eta_{t-1}$
- We can represent a linear process using the backward shift operator as $\eta_t = \mu + \psi(B)Z_t$, where we let $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$

Defining Linear Processes with Backward Shifts

- Recall the backward shift operator, B, is defined by $B\eta_t = \eta_{t-1}$
- We can represent a linear process using the backward shift operator as $\eta_t = \mu + \psi(B)Z_t$, where we let $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$
- Example: we can write a mean zero MA(1) process as

$$\eta_t = \mu + \psi(B)Z_t,$$

where $\mu = 0$ and $\psi(B) = 1 + \theta B$

Linear Filtering Preserves Stationarity

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- Let $\{Y_t\}$ be a time series and $\{\psi_j\}$ be a set of absolutely summable constants that does not depend on time
- **Definition**: A linear time invariant filtering of $\{Y_t\}$ with coefficients $\{\psi_j\}$ that do not depend on time is defined by

$$X_t = \psi(B)Y_t$$

- **Theorem**: Suppose $\{Y_t\}$ is a zero mean stationary series with ACVF $\gamma_Y(\cdot)$. Then $\{X_t\}$ is a zero mean stationary process with ACVF

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(j-k+h)$$

Example: The MA(q) Process is Stationary

By the filtering preserves stationarity result, the MA(q) process is a stationary process with mean zero and ACVF

$$\gamma(h) = \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}$$

Example: The MA(q) Process is Stationary

By the filtering preserves stationarity result, the MA(q) process is a stationary process with mean zero and ACVF

$$\gamma(h) = \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}$$

$$\begin{aligned}
\gamma(h) &= \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \gamma_Z(j-k+h) \\
&= \sigma^2 \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \mathbb{1}(k=j+h) \\
&= \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+h}
\end{aligned}$$

Processes with a Correlation that Cuts Off

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- A time series η_t is q-correlated if

  $$\eta_t \text{ and } \eta_s \text{ are uncorrelated } \forall |t-s| > q,$$

  i.e., $\text{Cov}(\eta_t, \eta_s) = 0, \forall |t-s| > q$

- A time series $\{\eta_t\}$ is q-dependent if

  $$\eta_t \text{ and } \eta_s \text{ are independent } \forall |t-s| > q.$$

- **Theorem**: if $\{\eta_t\}$ is a stationary q-correlated time series with zero mean, then it can be always be represented as an MA(q) process

AR(p): Autoregressive Process of Order p

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- This process is attributed to George Udny Yule. The AR(1) process has also been called the Markov process
- Let $\{Z_t\}$ be a $\mathrm{WN}(0, \sigma^2)$ process and let $\{\phi_1, \cdots, \phi_p\}$ be a set of constants for some integer $p > 0$ with $\phi_p \neq 0$
- The (zero-mean) AR(p) process is defined to be the solution to the equation

$$\eta_t = \sum_{j=1}^{p} \phi_j \eta_{t-j} + Z_t \Rightarrow \underbrace{\eta_t - \sum_{j=1}^{p} \phi_j \eta_{t-j}}_{\phi(B)\eta_t} = Z_t,$$

where we let $\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$

A Stationary Solution for AR(1)

- We want the solution to the AR equation to yield a stationary process. Let's first consider AR(1). We will demonstrate that **a stationary solution exists for** $|\phi_1| < 1$.

- We first write

$$\begin{aligned}
\eta_t = \phi_1 \eta_{t-1} + Z_t &= \phi_1(\phi_1 \eta_{t-2} + Z_{t-1}) + Z_t \\
&= \phi_1^2 \eta_{t-2} + \phi_1 Z_{t-1} + Z_t \\
&\vdots \\
&= \phi_1^k \eta_{t-k} + \sum_{j=0}^{k-1} \phi_1^j Z_{t-j} \\
&\vdots \\
&= \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}
\end{aligned}$$

AR(1) Example Cont'd

- Now let $\psi_j = \phi_1^j$. We then have

$$\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

  Using the fact that, for $|a| < 1$, $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$, the sequence $\{\psi_j\}$ is absolutely summable

- Thus, since $\{\eta_t\}$ is a linear process, it follows by the filtering preserves stationarity result that $\{\eta_t\}$ is a zero mean stationary process with ACVF

$$\begin{aligned}\gamma(h) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \\ &= \sigma^2 \sum_{j=0}^{\infty} \phi_1^j \phi_1^{j+h} \\ &= \sigma^2 \phi^h \sum_{j=0}^{\infty} (\phi_1^2)^j\end{aligned}$$

AR(1) Example Cont'd

Now $|\phi_1| < 1$ implies that $|\phi_1^2| < 1$ and therefore we have

$$\gamma(h) = \frac{\sigma^2 \phi_1^h}{1 - \phi_1^2}$$

When $|\phi_1| \geq 1$

- No stationary solutions exist for $|\phi_1| = 1$

AR(1) Example Cont'd

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$$\gamma(h) = \frac{\sigma^2 \phi_1^h}{1 - \phi_1^2}$$

When $|\phi_1| \geq 1$

- No stationary solutions exist for $|\phi_1| = 1$
- When $|\phi_1| > 1$, dividing by ϕ_1 for both sides we get

$$\phi_1^{-1}\eta_t = \eta_{t-1} + \phi_1^{-1} Z_t$$
$$\Rightarrow \eta_{t-1} = \phi_1^{-1}\eta_t - \phi_1^{-1} Z_t$$

A linear combination of **future** Z_t's $\Rightarrow$ we have a stationary solution, but, η_t depends on future $\{Z_t\}$'s–This process is said to be not causal

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  A linear combination of **future** Z_t's $\Rightarrow$ we have a stationary solution, but, η_t depends on future $\{Z_t\}$'s–This process is said to be not causal

- If we assume that η_s and Z_t are uncorrelated for each $t > s$, $|\phi_1| < 1$ is the only stationary solution to the AR equation

The Autoregressive Operator

- AR(1) process

$$\eta_t = \phi_1 \eta_{t-1} + Z_t \Rightarrow (1 - \phi_1 B)\eta_t = Z_t \Rightarrow \eta_t = (1 - \phi_1 B)^{-1} Z_t$$

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- Recall $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a} = (1-a)^{-1}$. We have

$$\eta_t = \sum_{j=0}^{\infty} (\phi_1 B)^j Z_t = \sum_{j=0}^{\infty} \phi_1^j B^j Z_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

$\Rightarrow$ **This is another way to show that AR(1) is a linear process**

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$\Rightarrow$ **This is another way to show that AR(1) is a linear process**

- Here $1 - \phi_1 B$ is the AR characteristic polynomial

The Second-Order Autoregressive Process: AR(2)

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Now consider the series satisfying

$$\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + Z_t,$$

where, again, we assume that Z_t is independent of $\eta_{t-1}, \eta_{t-2}, \cdots$

- The AR characteristic polynomial is

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

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The Second-Order Autoregressive Process: AR(2)

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$$\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + Z_t,$$

where, again, we assume that Z_t is independent of $\eta_{t-1}, \eta_{t-2}, \cdots$

- The AR characteristic polynomial is

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

- The corresponding AR characteristic equation is

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 = 0$$

Stationarity of the AR(2) Process

- A stationary solution exists if and only if the roots of the AR characteristic equation exceed 1 in absolute value
- For the AR(2) the roots of the quadratic characteristic equation are

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{-2\phi_2}$$

  These roots exceed 1 in absolute value if

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad \text{and } |\phi_2| < 1$$

- We say that the roots should lie outside the unit circle in the complex plane. This statement will generalize to the AR(p) case

The Autocorrelation Function for the AR(2) Process

- Yule-Walker equations:

$$
\begin{aligned}
\eta_t &= \phi_1 \eta_{t-1} + \phi_2 \eta_2 + Z_t \\
\Rightarrow \eta_t \eta_{t-h} &= \phi_1 \eta_{t-1} \eta_{t-h} + \phi_2 \eta_{t-2} \eta_{t-h} + Z_t \eta_{t-h} \\
\Rightarrow \gamma(h) &= \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) \\
\Rightarrow \rho(h) &= \phi_1 \rho(h-1) + \phi_2 \rho(h-2),
\end{aligned}
$$

$h = 1, 2, \cdots$

- Setting $h = 1$, we have $\rho(1) = \phi_1 \underbrace{\rho(0)}_{=1} + \phi_2 \underbrace{\rho(-1)}_{=\rho(1)} \Rightarrow \rho(1) = \frac{\phi_1}{1-\phi_2}$

- $\rho(2) = \phi_1 \rho(1) + \phi_2 \rho(0) = \frac{\phi_2(1-\phi_2)+\phi_1^2}{1-\phi_2}$

The Variance for the AR(2) Model

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Taking the variance of both sides of AR(2) equations:

$$\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + Z_t,$$

yields

$$\begin{aligned}
\gamma(0) &= \mathrm{Var}\left(\phi_1 \eta_{t-1} + \phi_2 \eta_{t-2}\right) + \mathrm{Var}(Z_t) \\
&= (\phi_1^2 + \phi_2^2)\gamma(0) + 2\phi_1\phi_2\gamma(1) + \sigma^2 \\
&= (\phi_1^2 + \phi_2^2)\gamma(0) + 2\phi_1\phi_2\left(\frac{\phi_1\gamma(0)}{1-\phi_2}\right) + \sigma^2 \\
&= \frac{(1-\phi_2)\sigma^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2) - 2\phi_2\phi_1^2} \\
&= \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma^2}{(1-\phi_2)^2 - \phi_1^2}
\end{aligned}$$

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The General Autoregressive Processes

Consider now the pth-order autoregressive model:

$$\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + \cdots + \phi_p \eta_{t-p} + Z_t$$

- AR characteristic polynomial:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

  AR characteristic equation:

$$1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p = 0$$

- Yule-Walker equations:

$$\begin{aligned}
\rho(1) &= \phi_1 + \phi_2 \rho(1) + \cdots + \phi_p \rho(p-1) \\
\rho(2) &= \phi_1 \rho(1) + \phi_2 + \cdots + \phi_p \rho(p-2) \\
&\vdots \\
\rho(p) &= \phi_1 \rho(p-1) + \phi_2 \rho(p-2) + \cdots + \phi_p
\end{aligned}$$

- Variance:

$$\begin{aligned}
\gamma(0) &= \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \sigma^2 \\
&= \frac{\sigma^2}{1 - \phi_1 \rho(1) - \cdots - \phi_p \rho(p)}
\end{aligned}$$

ARMA(p, q) Processes

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Partial Autocorrelation Functions

$\{\eta_t\}$ is an ARMA(p, q) process if it satisfies

$$\eta_t - \sum_{i=1}^{p} \phi_i \eta_{t-i} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where $\{Z_t\}$ is a $\mathrm{WN}(0, \sigma^2)$ process.

- Let $\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ and $\theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$. Then we can write it as

$$\phi(B)\eta_t = \theta(B) Z_t$$

- An ARMA(p, q) process $\{\tilde{\eta}_t\}$ with mean μ can be written as

$$\phi(B)(\tilde{\eta}_t - \mu) = \theta(B) Z_t$$

A Stationary Solution to the ARMA Equation

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A zero-mean ARMA process is stationary if it can be written as a linear process, i.e., $\eta_t = \psi(B)Z_t$, where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$ for an absolutely summable sequence $\{\psi_j\}$

- This only happens if one can "divide" by $\phi(B)$, i.e., it is stationary only if the following makes senese:

$$(\phi(B))^{-1}\phi(B)\eta_t = (\phi(B))^{-1}\theta(B)Z_t$$

- Let's forget about B is the backshift operator and replace it with z. Now consider whether we can divide $\theta(z)$ by $\phi(z)$

The Roots of AR Characteristic Polynomial and Stationarity

- A root of the polynomial $f(z) = \sum_{j=0}^{p} a_j z^j$ is a value ξ such that $f(\xi) = 0 \Rightarrow$ it can be real-valued $\mathbb{R}$ or complex-valued $\mathbb{C}$
- For example, a root can take the form $\xi = a + b\,i$ for real number a and b. The modulus of a complex number $|\xi|$ is defined by

$$|\xi| = \sqrt{a^2 + b^2}$$

- For any ARMA(p,q) process, a stationary and unique solution exists if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| = 1$.

Note: Stationarity of the ARMA process has nothing to do with the MA polynomial!

AR(4) Example

Consider the following AR(4) process

$$\eta_t = 2.7607\eta_{t-1} - 3.8106\eta_{t-2} + 2.6535\eta_{t-3} - 0.9238\eta_{t-4} + Z_t,$$

the AR characteristic polynomial is

$$\phi(z) = 1 - 2.7607z + 3.8106z^2 - 2.6535z^3 + 0.9238z^4$$

- Hard to find the roots of $\phi(z)$ –we use the `polyroot` function in `R`:
- Use `Mod` in `R` to calculate the modulus of the roots
- **Conclusion:**

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Causal ARMA Processes

An ARMA process is causal if there exists constants $\{\psi_j\}$ with $\sum_{j=0}^{\infty} |\psi_j| < 0$ and $\eta_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, that is, we can write $\{\eta_t\}$ as an MA(∞) process depending only on the current and past values of $\{Z_t\}$

- Equivalently, an ARMA process is causal if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0,$$

for all $|z| \leq 1$

- The previous AR(4) example is causal since each zero, ξ, of $\phi(\cdot)$ is such that $|\xi| > 1$

Invertible ARMA Processes

An ARMA process is invertible if there exists constants $\{\pi_j\}$ with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j \eta_{t-j},$$

that is, we can write $\{Z_t\}$ as an AR(∞) process depending only on the current and past values of $\{\eta_t\}$

- A process is invertible if and only if

  $$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0,$$

  for all $|z| \leq 1$

- An ARMA process

  $$\phi(B)\eta_t = \theta(B)Z_t,$$

  with $\phi(z) = 1 - 0.5z$ and $\theta(z) = 1 + 0.4z$ has a root of the MA characteristic polynomial at $z = \frac{-1}{0.4} = -2.5$

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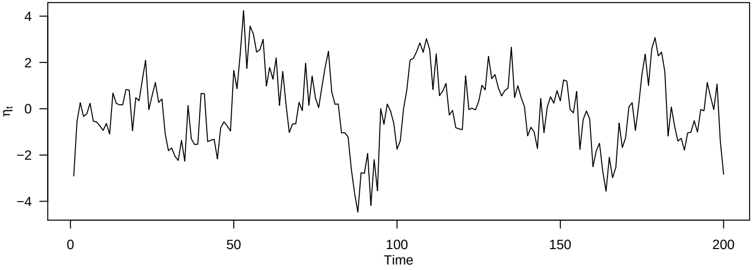
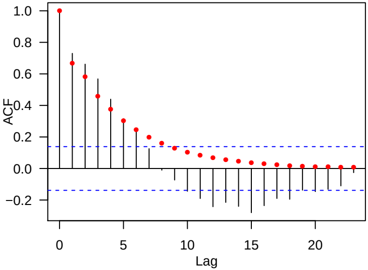
Partial Autocorrelation Functions (PACF)

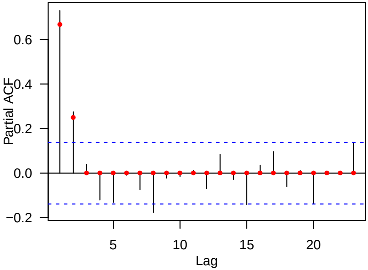
The partial autocorrelation function (PACF) represents the partial correlation of a stationary time series $\{\eta_t\}$ with its own lagged values, while regressing out the effects of the time series at all shorter lags

- PACF of lag h is the autocorrelation between η_t and η_{t+h} with the linear dependence between η_t and $\eta_{t+1}, \cdots, \eta_{t+h-1}$ removed
- PACF plots are a commonly used tool for identifying the order of an AR model, as the theoretical PACF "shuts off" past the order of the model
- One can use the function `pacf` in `R` to plot the PACF plots

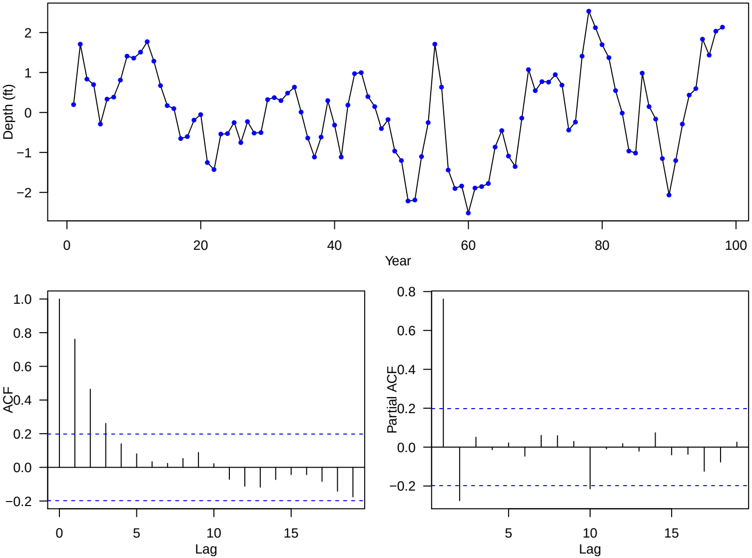
An Example of PACF Plot



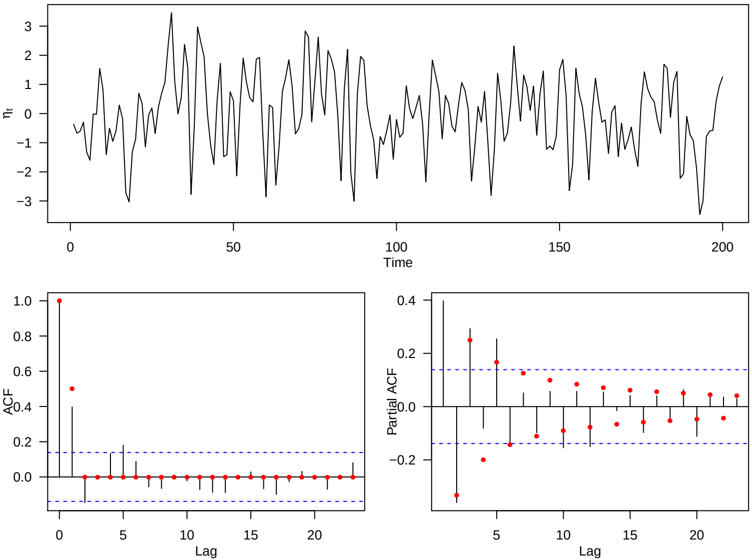




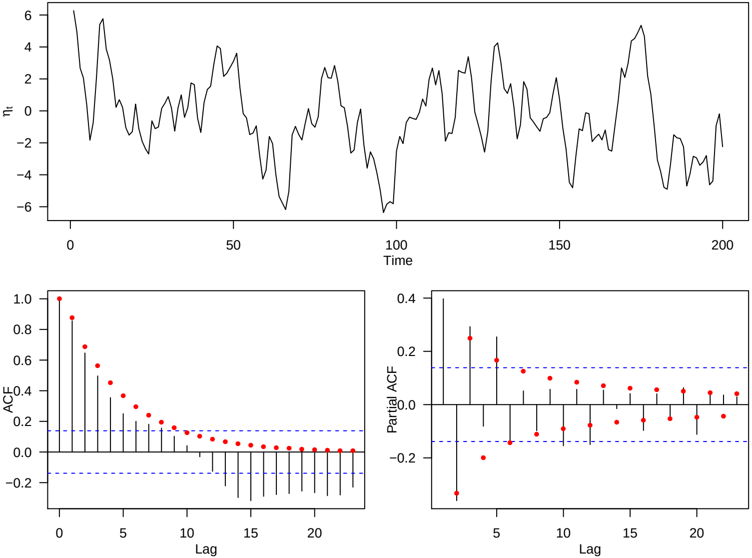
Lake Huron Series PACF Plot



PACF Plot for a MA Process



PACF Plot for a ARMA Process



Identifying Plausible Stationary ARMA Models

We can use the sample ACF and PACF to help identify plausible models:

Model	ACF	PACF
MA(q)	cuts off after lag q	tails off exponentially
AR(p)	tails off exponentially	cuts off after lag p

For ARMA(p, q) we will see a combination of the above