Lecture 6

Probability II

Readings: IntroStat Chapter 4; OpenIntro Chapter 3

STAT 8010 Statistical Methods I

May 23, 2023

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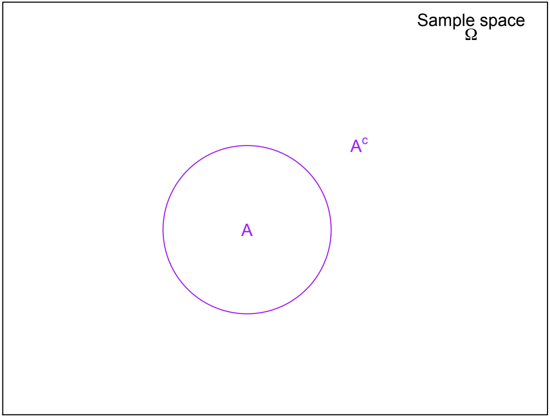
Agenda

1. **Complement Rule and General Addition Rule**
2. **Independence and Conditional Probability**
3. **Law of Total Probability**
4. **Bayes' Rule**

Complement Rule and General Addition Rule

Complement



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4. Hence we get $\boxed{\mathbb{P}(A) = 1 - \mathbb{P}(A^c)}$

Example

Suppose we rolled a fair, six–sided die 10 times. Let T be the event that we roll at least 1 three. If one were to calculate T you would need to find the probability of 1 three, 2 threes, $\cdots$, and 10 threes and add them all up. However, you can use the complement rule to calculate $\mathbb{P}(T)$

Solution.

Let X be the times that we rolled a 3, then

$$\mathbb{P}(T) = \mathbb{P}(X \geq 1) = \underbrace{\mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \cdots + \mathbb{P}(X = 10)}_{\text{need to compute 10 probabilities}}$$

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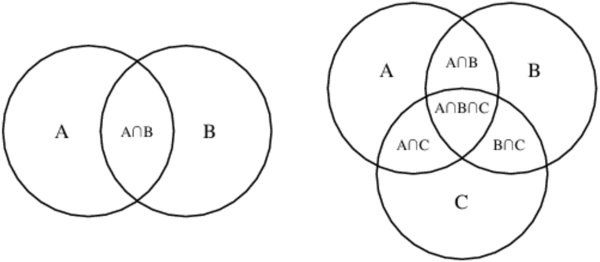
If we apply the complement rule

$$\mathbb{P}(T) = 1 - \mathbb{P}(T^c) = 1 - \mathbb{P}(X = 0)$$

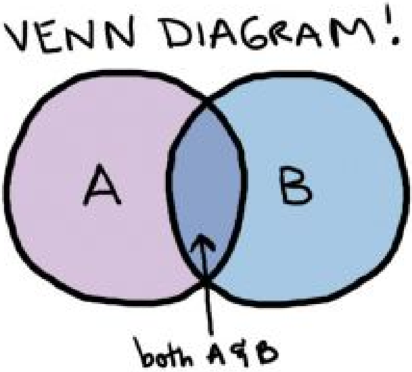
Venn Diagram

A Venn diagram is a diagram that shows all possible logical relations between a finite collection of events.



General Addition Rule

The general addition rule is a way of finding the probability of a union of 2 events. It is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



Example

Three of the major commercial computer operating systems are Windows, Mac OS, and Red Hat Linux Enterprise. A Computer Science professor selects 50 of her students and asks which of these three operating systems they use. The results for the 50 students are summarized below.

- 30 students use Windows
- 16 students use at least two of the operating systems
- 9 students use all three operating systems
- 18 students use Mac OS
- 46 students use at least one of the operating systems
- 11 students use both Windows and Linux
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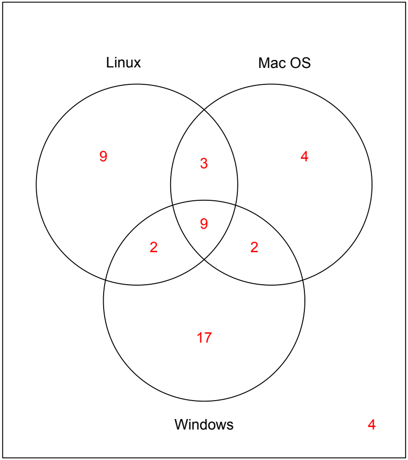
Example cont'd



Independence and Conditional Probability

Independence: A Motivating Example

Example

You toss a fair coin and it comes up “Heads" three times. What is the chance that the next toss will also be a “Head"?

Independence and Conditional Probability

Conditional Probability

Let A and B be events. The probability that event B occurs given (knowing) that event A occurs is called a conditional probability and is denoted by $P(B|A)$. The formula of conditional probability is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Independent events

Suppose $P(A) > 0,\ P(B) > 0$. We say that event B is independent of event A if the occurrence of event A does not affect the probability that event B occurs.

$$P(B|A) = P(B) \Rightarrow P(B \cap A) = P(B)P(A)$$

Law of Partitions & Multiplication Rule

Law of partitions

Let $A_1, A_2, \cdots, A_k$ form a partition of Ω. Then, for all events B,

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(A_i \cap B)$$

Multiplication rule

- 2 events:

$$\mathbb{P}(B \cap A) = \mathbb{P}(A) \times \mathbb{P}(B|A) = \mathbb{P}(B) \times \mathbb{P}(A|B)$$

- More than 2 events:

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_1 \cap A_2) \times \cdots \times \mathbb{P}(A_n|A_{n-1} \cap \cdots \cap A_1)$$

Law of Total Probability

Let $A_1, A_2, \cdots, A_k$ form a partition of Ω. Then, for all events B,

$$\mathbb{P}(B) = \underbrace{\sum_{i=1}^{k} \mathbb{P}(A_i \cap B)}_{\text{Law of partitions}}$$

$$= \underbrace{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}_{\text{Multiplication rule}}$$

Example

Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

Bayes' Rule: Motivating example

The Monty Hall Problem

There was an old television show called Let's Make a Deal, whose original host was named Monty Hall. The set–up is as follows. You are on a game show and you are given the choice of three doors. Behind one door is a car, behind the others are goats. You pick a door, and the host, who knows what is behind the doors, opens another door (not your pick) which has a goat behind it. Then he asks you if you want to change your original pick. The question we ask you is, "Is it to your advantage to switch your choice?"

The Monty Hall Problem

The Monty Hall Problem Solution

Bayes' Rule

General form

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Let $A_1, A_2, \cdots, A_k$ form a partition of the sample space. Then for every event B in the sample space,

$$\boxed{\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \times \mathbb{P}(A_j)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i) \times \mathbb{P}(A_i)}}, \; j = 1, 2, \cdots, k$$

Example

Let us assume that a specific disease is only present in 5 out of every 1,000 people. Suppose that the test for the disease is accurate 99% of the time a person has the disease and 95% of the time that a person lacks the disease. What is the probability that the person has the disease given that they tested positive?

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Solution.

$$\mathbb{P}(D|+) = \frac{\mathbb{P}(D\cap+)}{\mathbb{P}(+)} = \frac{.005\times.99}{.005\times.99+.995\times.05} = \frac{.00495}{.0547} = .0905$$

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The reason we get such a surprising result is because the disease is so rare that the number of false positives greatly outnumbers the people who truly have the disease.

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- Independence: if A and B are independent, then
  $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(B|A) = \mathbb{P}(B)$, and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$