# Lecture 7

## Probability III

Readings: IntroStat Chapter 4; OpenIntro Chapter 3

*STAT 8010 Statistical Methods I*
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Random Variables
Bernoulli and Binomial Random Variables
Hypergeometric Random Variable

# Agenda

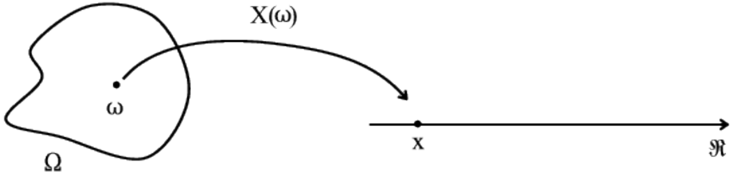
1. **Random Variables**
2. **Bernoulli and Binomial Random Variables**
3. **Hypergeometric Random Variable**

# Random Variables

A random variable is a real–valued function whose domain is the sample space of a random experiment. In other words, a random variable is a function

$$X : \Omega \mapsto \mathbb{R}$$

where $\Omega$ is the sample space of the random experiment under consideration and $\mathbb{R}$ represents the set of all real numbers.



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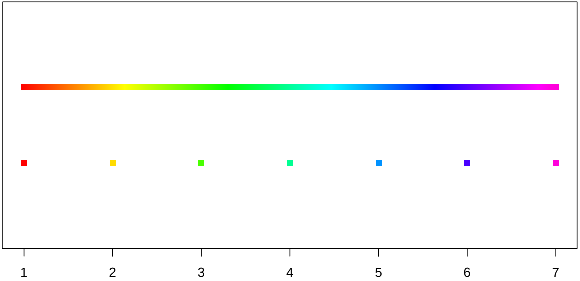
# Discrete and Continuous Random Variables

There are two main types of quantitative random variables (r.v.s): discrete and continuous. A discrete r.v. often involves a count of something.

**Discrete random variable**

A random variable $X$ is called a discrete random variable if the outcome of the random variable is limited to a countable set of real numbers (usually integers).



## Example

The following is a chart describing the number of siblings each student in a particular class has.

| Siblings | Frequency | Relative Frequency |
|---|---|---|
| 0 | 8 | .200 |
| 1 | 17 | .425 |
| 2 | 11 | .275 |
| 3 | 3 | .075 |
| 4 | 1 | .025 |
| Total | 40 | 1 |

Let's define the event $A$ as the event that a randomly chosen student has 2 or more siblings. What is $\mathbb{P}(A)$?

**Solution.**

$$\mathbb{P}(A) = \mathbb{P}(X \geq 2) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4)$$
$$= .275 + .075 + .025 = .375$$

# Probability Mass Function



Let $X$ be a discrete random variable. Then the probability mass function (pmf) of $X$ is the real–valued function defined on $\mathbb{R}$ by

$$p_X(x) = \mathbb{P}(X = x)$$

The capital letter, $X$, is used to denote random variable. Lowercase letter, $x$, is used to denote possible values of the random variable.

$p_X(x)$: The probability that the discrete random variable $X$ is exactly equal to $x$.

# Probability Mass Function Example



Random Variables
Bernoulli and Binomial Random Variables
Hypergeometric Random Variable

Flip a fair coin 3 times. Let $X$ denote the number of heads tossed in the 3 flips. Create a pmf for $X$

**Solution.**

The random variable $X$ maps any outcome to an integer (e.g. $X((\text{T, T, T})) = 0, X((H, H, T)) = 2$)

## Probability Mass Function Example

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| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_X(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

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# Properties of a PMF



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- $0 \le p_X(x) \le 1, \; x \in \{0, 1, 2, \cdots\}$
- $\sum_x p_X(x) = 1$

# Example

Let $X$ be a random variable with pmf defined as follows:

$$p_X(x) = \begin{cases} k(5-x) & \text{if } \; x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the value of $k$ that makes $p_X(x)$ a legitimate pmf.
2. What is the probability that $X$ is between 1 and 3 inclusive?
3. If $X$ is not 0, what is the probability that $X$ is less than 3?

## Mean of Discrete Random Variables

The mean of a discrete r.v. $X$, denoted by $\mathbb{E}[X]$, is defined by

$$\mathbb{E}[X] = \sum_x x \times p_X(x)$$

**Remark:**

The mean of a discrete r.v. is a weighted average of its possible values, and the weight used is its probability. Sometimes we refer to the expected value as the expectation (expected value), or the first moment.

For any function, say $g(X)$, we can also find an expectation of that function. It is

$$\mathbb{E}[g(X)] = \sum_x g(x) \times p_X(x)$$

**Example**

$$\mathbb{E}[X^2] = \sum_x x^2 \times p_X(x)$$

# Properties of Mean

Let $X$ and $Y$ be discrete r.v.s defined on the same sample space and having finite expectation (i.e. $\mathbb{E}[X], \mathbb{E}[Y] < \infty$). Let $a$ and $b$ be constants. Then the following hold:

- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

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- $\mathbb{E}[aX + b] = a \times \mathbb{E}[X] + b$

# Number of Siblings Example Revisited







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Find the expected value of the number of siblings

**Solution.**

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Find the expected value of the number of siblings

**Solution.**

$$\mathbb{E}[X] = \sum_x x p_X(x) = 0 \times .200 + 1 \times .425 + 2 \times .275 + 3 \times .075 + 4 \times .025 = 1.3$$

# Variance/Standard Deviation of Discrete r.v.'s





The **variance** of a (discrete) r.v., denoted by $Var(X)$, is a measure of the spread, or variability, in the r.v. $Var(X)$ is defined by

$$\boxed{Var(X) = \mathbb{E}[(X - \mathbb{E}[x])^2]}$$

or

$$\boxed{Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

The **standard deviation**, denoted by $sd(X)$, is the square root of its variance

# Properties of Variance



Let $c$ be a constant. Then the following hold:

- $Var(cX) = c^2 \times Var(X)$

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Let $c$ be a constant. Then the following hold:

- $Var(cX) = c^2 \times Var(X)$
- $Var(X + c) = Var(X)$

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## Example

Suppose $X$ and $Y$ are random variables with $\mathbb{E}[X] = 3$, $\mathbb{E}[Y] = 4$ and $Var(X) = 4$. Find:

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4. $\mathbb{E}[X^2 - 4]$
5. $\mathbb{E}[(X - 4)^2]$

# Example



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1. $\mathbb{E}[2X + 1]$
2. $\mathbb{E}[X - Y]$
3. $\mathbb{E}[X^2]$
4. $\mathbb{E}[X^2 - 4]$
5. $\mathbb{E}[(X - 4)^2]$
6. $Var(2X - 4)$

## Bernoulli Trials

Many problems in probability and its applications involve independently repeating a random experiment and observing at each repetition whether a specified event occurs. We label the occurrence of the specified event a success and the nonoccurrence of the specified event a failure.

**Example:**
Tossing a coin several times

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# Bernoulli Trials Cont'd

Bernoulli trials:

- Each repetition of the random experiment is called a trial
- We use $p$ to denote the probability of a success on a single trial

Properties of Bernoulli trials:

- Exactly two possible outcomes success and failure
- The outcomes of trials are independent of one another
- The success probability, $p$, and therefore the failure probability, $(1 - p)$, remains the same from trial to trial

## Binomial Random Variable

We define the Binomial r.v. as the number of successes in $n$ Bernoulli trials, where the probability of success in one trial is $p$. Let $X$ be a Binomial r.v.

- The definition of $X$: # of successes in $n$ trials of Bernoulli trials.

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$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, \cdots, n$$

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- The expected value:

$$\mathrm{E}[X] = np$$

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- The expected value:

$$\mathrm{E}[X] = np$$

- The variance:

$$\mathrm{Var}(X) = np(1-p)$$

# Example

To test for Extrasensory perception (ESP), we have 4 cards. They will be shuffled and one randomly selected each time, and you are to guess which card is selected. This is repeated 10 times. Suppose you do not have ESP. Let $R$ be the number of times you guess a card correctly. What are the distribution and parameter(s) of $R$? What is the expected value of $R$? Furthermore, suppose that you get certified as having ESP if you score at least an 8 on the test. What is the probability that you get certified as having ESP?

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**Solution.**

$$R \sim Binomial(n = 10, p = 1/4 = .25)$$
$$\mathrm{E}[R] = n \times p = 2.5$$
$$\mathrm{P}(X \geq 8) = .000416$$

## Example

Suppose that 95% of consumers can recognize Coke in a blind taste test. Assume consumers are independent of one another. The company randomly selects $4$ consumers for a taste test. Let $X$ be the number of consumers who recognize Coke.

1. What is the probability that $X$ is at least 1?
2. What is the probability that $X$ is at most 3?

# Binomial and Hypergeometric r.v.s

The binomial distribution describes the probability of $k$ successes in $n$ trials with replacement.

We want a distribution to describe the probability of $k$ successes in $n$ trials without replacement from a finite population of size $N$ containing exactly $K$ successes.

$\Rightarrow$ Hypergeometric Distribution

Important applications are **quality control** and statistical **estimation of population proportions**. The hypergeometric r.v. is the equivalent of a Binomial r.v. except that sampling is done without replacement.

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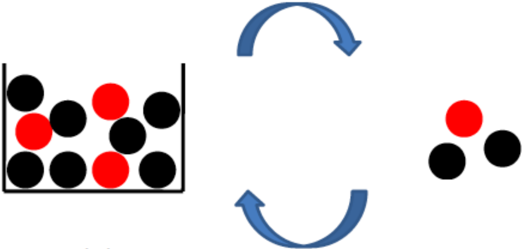
# An Example of Hypergeometric r.v.

Probability:

What is the probability to get 1 red and 2 black balls?



Statistics:

What percentage of balls in the box are red?

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# Hypergeometric r.v.s

Let $X$ be a hypergeometric r.v.

- The definition of $X$: # of successes in $n$ trials of a random experiment, where sampling is done without replacement (or trials are dependent)
- The support: $k \in \{\max(0, n+K-N), \cdots, \min(n, K)\}$
- Its parameter(s) and definition(s): $N$: the population size, $n$: the sample size, and $K$: number of success in the population
- The probability mass function (pmf): $p_X(k) = \frac{\binom{K}{k} \times \binom{N-K}{n-k}}{\binom{N}{n}}$
- The expected value: $\mathrm{E}[X] = n\frac{K}{N}$
- The variance: $\mathrm{Var}(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$

# Example



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There are 100 identical looking 52" TVs at Best Buy in Anderson, SC. Let 10 of them be defective. Suppose we want to buy 8 of the aforementioned TVs (at random). What is the probability that we don't get any defective TVs?

**Solution.**

## Example

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**Solution.**

Let $D$ be the number of defective TVs in the sample.

$D \sim Hyp(N = 100, n = 8, K = 10)$

$\mathrm{P}(D = 0) = \frac{\binom{10}{0}\binom{90}{8}}{\binom{100}{8}} = 0.4166$

# Summary

In this lecture, we learned

- Random Variables
- The probability mass function, mean, and variance of a discrete random variable
- Examples of discrete random variables: Bernoulli/Binomial, Hyper-geometric