

# Lecture 22

## Completely Randomized Designs

STAT 8020 Statistical Methods II  
November 12, 2020

Whitney Huang  
Clemson University

Completely Randomized Designs



22.1

Notes

---

---

---

---

---

---

---

---

### Completely Randomized Designs

A completely randomized design (CRD) has

- $g$  different treatments
- $g$  known treatment group sizes  $n_1, n_2, \dots, n_g$  with  $\sum_{i=1}^g n_i = N$
- Completely random assignment of treatments to the experimental units

This is the basic experimental design; everything else is a modification

- Easiest to analyze
- Most resilient when things go wrong
- Often sufficient

Completely Randomized Designs



22.2

Notes

---

---

---

---

---

---

---

---

### Inference

- Any evidence means (i.e.,  $\{\mu_1, \mu_2, \dots, \mu_g\}$ ) are not all the same?  $\Rightarrow$  ANOVA
- Which ones differ?  $\Rightarrow$  Multiple comparisons
- Estimates/confidence intervals of means and differences

Completely Randomized Designs



22.3

Notes

---

---

---

---

---

---

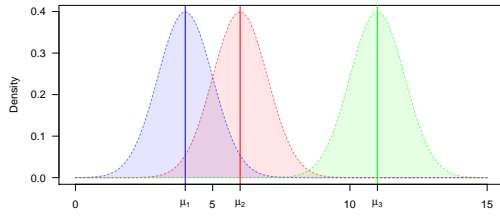
---

---

**Statistical Model: Means Model**

Let  $Y_{ij}$  be the random variable that represents the response for the  $j^{\text{th}}$  experimental unit to treatment  $i$ . Let  $\mu_i = E(Y_{ij})$  be the mean response for the  $i^{\text{th}}$  treatment. We have

$$Y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, \dots, g, \quad j = 1, \dots, n_i, \quad \epsilon_{ij} \sim N(0, \sigma^2)$$



Notes

---

---

---

---

---

---

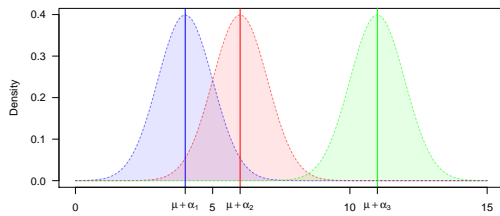
---

---

**Effects Model**

Alternatively, we could let  $\mu_i = \mu + \alpha_i$ , which leads to

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad i = 1, \dots, g, \quad j = 1, \dots, n_i, \quad \epsilon_{ij} \sim N(0, \sigma^2)$$



**Overparameterized.** Need to add a constraint so that the parameters are estimable.

Notes

---

---

---

---

---

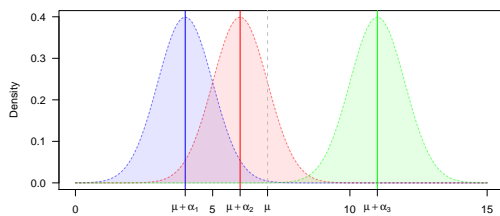
---

---

---

**Effects Model Cont'd**

Suppose we let  $\sum_{i=1}^g n_i \alpha_i = 0$



Notes

---

---

---

---

---

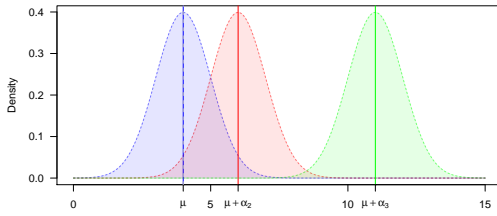
---

---

---

## Effects Model Cont'd

Suppose we let  $\alpha_1 = 0$



Notes

---

---

---

---

---

---

---

---

## Data Layout & the Dot Notation

$y_{ij}$  is the "observed" response for the  $j^{\text{th}}$  experimental unit to treatment  $i$ .

Treatment	Observations				Totals	Averages
1	$y_{11}$	$y_{12}$	$\cdots$	$y_{1n_1}$	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	$y_{21}$	$y_{22}$	$\cdots$	$y_{2n_2}$	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$g$	$y_{g1}$	$y_{g2}$	$\cdots$	$y_{gn_g}$	$y_{g\cdot}$	$\bar{y}_{g\cdot}$
				$y_{\cdot\cdot}$	$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

Notes

---

---

---

---

---

---

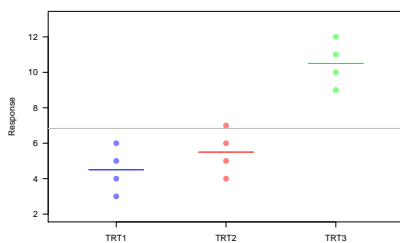
---

---

## ANOVA

Decomposition of  $y_{ij}$ :  $y_{ij} = \bar{y}_{\cdot\cdot} + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + (y_{ij} - \bar{y}_{i\cdot})$

$$\Rightarrow \underbrace{\sum_{i=1}^g \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{\cdot\cdot})^2}_{SS_T} = \underbrace{\sum_{i=1}^g n_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2}_{SS_{TRT}} + \underbrace{\sum_{i=1}^g \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i\cdot})^2}_{SS_E}$$



Notes

---

---

---

---

---

---

---

---

## ANOVA Table

Source	df	SS	MS	EMS
Treatment	$g - 1$	$SS_{TRT}$	$MS_{TRT} = \frac{SS_{TRT}}{g-1}$	$\sigma^2 + \frac{\sum_{i=1}^g n_i \alpha_i^2}{g-1}$
Error	$N - g$	$SS_E$	$MS_E = \frac{SS_E}{N-g}$	$\sigma^2$
Total	$N - 1$	$SS_T$		

$$SS_T = \sum_{i=1}^g \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^g \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$SS_{TRT} = \sum_{i=1}^g n_i (\bar{y}_{i.} - \bar{y}_{..})^2 = \sum_{i=1}^g \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N}$$

$$SS_E = \sum_{i=1}^g \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 = \sum_{i=1}^g \sum_{j=1}^{n_i} y_{ij}^2 - \sum_{i=1}^g \frac{y_{i.}^2}{n_i} = SS_T - SS_{TRT}$$

Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.10

Notes

---

---

---

---

---

---

---

---

---

---

## F-Test

### Testing for treatment effects

$$H_0 : \alpha_i = 0 \quad \text{for all } i$$

$$H_a : \alpha_i \neq 0 \quad \text{for some } i$$

**Test statistics:**  $F = \frac{MS_{TRT}}{MS_E}$ . Under  $H_0$ , the test statistic follows an F-distribution with  $g - 1$  and  $N - g$  degrees of freedom

Reject  $H_0$  if

$$F_{obs} > F_{g-1, N-g; \alpha}$$

for an  $\alpha$ -level test,  $F_{g-1, N-g; \alpha}$  is the  $100 \times (1 - \alpha)\%$  percentile of a **central F-distribution** with  $g - 1$  and  $N - g$  degrees of freedom.

The **P-value** of the F-test is the probability of obtaining  $F$  at least as extreme as  $F_{obs}$ , that is,  $P(F > F_{obs}) \Rightarrow$  reject  $H_0$  if P-value  $< \alpha$ .

Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.11

Notes

---

---

---

---

---

---

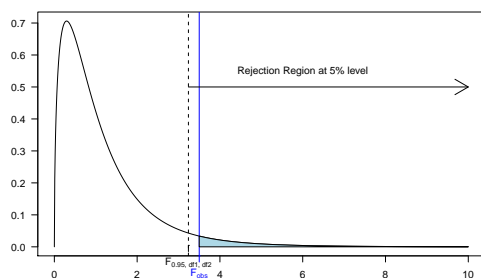
---

---

---

---

## F Distribution and the F-Test



Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.12

Notes

---

---

---

---

---

---

---

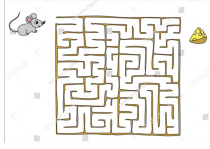
---

---

---

### Example

An experiment was conducted to determine if experience has an effect on the time it takes for mice to run a maze. Four treatment groups, consisting of mice having been trained on the maze one, two, three and four times were run through the maze and their times recorded.



Source: <https://www.shutterstock.com/image-vector/find-your-way-cheese-mouse-maze-232569073>

Training runs	1	2	3	4
$n_i$	5	5	5	5
$\bar{y}_i$	9.14	7.24	6.76	5.18
$s_i^2$	0.308	0.418	0.313	0.262

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.13

Notes

---

---

---

---

---

---

---

---

### Example Cont'd

Training runs	1	2	3	4
$n_i$	5	5	5	5
$\bar{y}_i$	9.14	7.24	6.76	5.18
$s_i^2$	0.308	0.418	0.313	0.262

- Write down the model.
- Fill out the ANOVA table and test whether the time to run the maze is affected by training. Use a significant level of .05.

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.14

Notes

---

---

---

---

---

---

---

---

### Model Assumptions

**Model:**

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad i = 1, \dots, g, \quad j = 1, \dots, n_i.$$

We make the following **assumptions**:

- Errors normally distributed
- Errors have constant variance
- Errors are independent

$$\Rightarrow \epsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$$

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.15

Notes

---

---

---

---


---

---

---

---

*All models are wrong  
but some are useful*



George E.P. Box

Notes

---

---

---

---

---

---

---

---

### What If Assumptions are Violated?

If the assumptions are not true, our statistical inferences might not be valid, for example,

- A confidence interval might not cover with the stated coverage rate
- A test with nominal type I error could actually have a larger or smaller type I error rate

We need good strategy for checking model assumptions, i.e.,  $\epsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$ .

Notes

---

---

---

---

---

---

---

---

### Checking Model Assumptions

We need to check if these assumptions reasonably met

**Model:**

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

**Data:**

$y_{ij}$	=	$(\bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}))$	+	$(y_{ij} - \bar{y}_{i.})$
$y_{ij}$	=	$\hat{y}_{ij}$	+	$\hat{\epsilon}_{ij}(r_{ij})$
observed	=	predicted	+	residual

Residuals are our "estimates" of unobservable errors  $\epsilon'_{ij,s}$

We will conduct model diagnostics using **residual** and **predicted** values.

Notes

---

---

---

---

---

---

---

---

## Residuals

We will use residuals to assess the model assumptions.

- **Raw residual:**

$$r_{ij} = y_{ij} - \hat{y}_{ij}, \text{ where } \hat{y}_{ij} = \hat{\mu} + \hat{\alpha}_i = \bar{y}_i.$$

- **Standardized residual (internally Studentized residual)** adjusts  $r_{ij}$  for its estimated standard deviation

$$s_{ij} = \frac{r_{ij}}{\sqrt{MS_E(1 - \frac{1}{n_i})}}$$

- **Studentized residual (externally Studentized residual)**

$$t_{ij} = s_{ij} \sqrt{\frac{N - g - 1}{N - g - s_{ij}^2}}$$

$t_{ij} \sim t_{df=N-g-1}$  if the model is correct  $\Rightarrow$  can be used to identify **outliers**



Notes

---

---

---

---

---

---

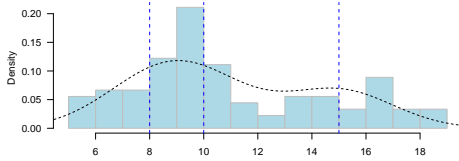
---

---

## Assessing Normality

We DO NOT assume all  $y'_{ij}$ s come from the same normal distribution, instead we assume  $e'_{ij}$ s come from the same normal distribution  $\Rightarrow$  Not informative to plot a histogram for all the data—treatment effects lead to non-normality

**Example:** Suppose  $g = 3$ ,  $(\mu_1, \mu_2, \mu_3) = (8, 10, 15)$  and  $e'_{ij;s} \sim N(0, 2^2)$



Notes

---

---

---

---

---

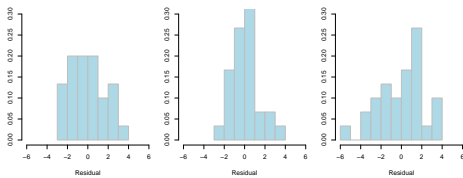
---

---

---

## Assessing Normality Cont'd

- If sample sizes are large, histograms of **residuals** can be constructed from each treatment separately



- Also, if sample sizes are large, QQ-plots or normal quantile plots can be generated for each treatment



Notes

---

---

---

---

---

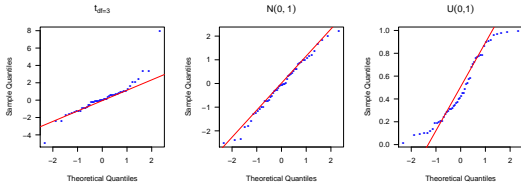
---

---

---

## Normal Quantile Plots

Plots  $r_{(k)}$  versus  $\Phi^{-1}\left(\frac{k-3/8}{n+1/4}\right)$ ,  $k = 1, \dots, n$ , where  $r_{(k)}$  is the  $k^{\text{th}}$  ordered residual and  $\Phi^{-1}\left(\frac{k-3/8}{n+1/4}\right)$  is its corresponding (standard) normal score.



Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.22

Notes

---

---

---

---

---

---

---

---

---

---

## Remarks on Assessing Normality

- Assessing normality
  - Formal tests (e.g., Shapiro–Wilk test, Anderson–Darling test) are usually not useful:
    - With small sample sizes, one will never be able to reject  $H_0$ , with large sample sizes, one will constantly detect little deviations that have no practical effect
  - Assess normal assumption graphically using QQ-plots or histograms
- Dealing with Non-normality
  - Use non-parametric procedure such as Kruskal–Wallis test (1952)
  - Transformation such as Box-Cox (1964)
- F-test is robust to non-normality

Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.23

Notes

---

---

---

---

---

---

---

---

---

---

## Assessing Equal Variance

- We can test for equal variance, but some tests rely heavily on normality assumption:
  - Hartley's test
  - Bartlett's test
  - Cochran's C test
- F-test is reasonably robust to unequal variance if  $n_i^2 s_i^2$  are equal, or nearly so
- "If you have to test for equality of variances, your best bet is Levene's test." – Gary Oehlert

Completely  
Randomized  
Designs

CLEMSON  
UNIVERSITY

22.24

Notes

---

---

---

---

---

---

---

---

---

---



### Levene's Test

- 1 Compute  $r_{ij} = y_{ij} - \bar{y}_i$ .
- 2 Treat the  $|r_{ij}|$  as data and use the ANOVA F-test to test  $H_0$  that the groups have the same average value of  $|r_{ij}|$
- 3 If  $\frac{MS_{TRT}}{MS_E} > F_{g-1, N-g-1; \alpha} \Rightarrow$  reject  $H_0$
- 4 Modified Levene's (Brown-Forsythe) test: use  $d_{ij} = |y_{ij} - \tilde{y}_i|$ , the absolute deviations from the group medians instead of  $|r_{ij}|$

Fairly robust to non-normality and unequal sample size

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.25

### Notes

---

---

---

---

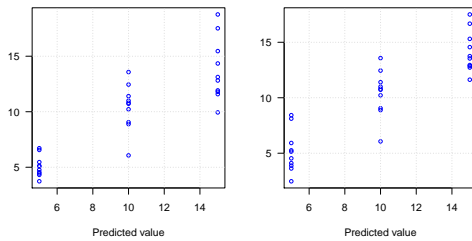
---

---

---

---

### Diagnostic Plot for Non-Constant Variance



Use this residual versus predicted value (treatment) plot to assess equal variance assumption and search for possible outliers

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.26

### Notes

---

---

---

---

---

---

---

---

### Remarks on Assessing Constant Variance Assumption

- Checking constant variance assumption: Assess the assumption qualitatively, don't just rely on tests
- Dealing with unequal variance
  - Variance-stabilizing transformations
  - Account unequal variance in the model
- F-test is reasonably robust to unequal variance if we have (nearly) balanced designs

Completely Randomized Designs  
CLEMSON UNIVERSITY  
22.27

### Notes

---

---

---

---

---

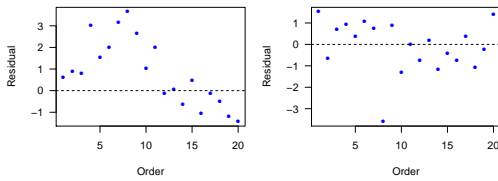
---

---

---

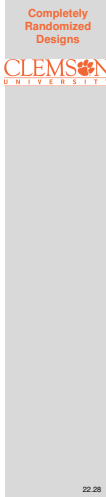
## Assessing Dependence

Independence is often argued via randomization. However, plotting residuals versus **run order** or **spatial location** can give information on lack of independence.



**Durbin–Watson statistic** is a simple numerical method for checking serial dependence:

$$DW = \frac{\sum_{k=1}^{n-1} (r_k - r_{k+1})^2}{\sum_{k=1}^n r_k^2}$$



Notes

---

---

---

---

---

---

---

---

---

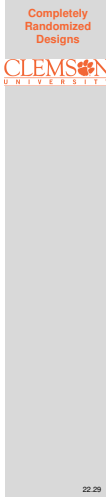
---

## Example: Balloon Experiment (taken from Dean and Voss Exercise 3.12)

The experimenter (Meily Lin) had observed that some colors of birthday balloons seem to be harder to inflate than others. She ran this experiment to determine whether balloons of different colors are similar in terms of the time taken for inflation to a diameter of 7 inches. Four colors were selected from a single manufacturer. An assistant blew up the balloons and the experimenter recorded the times with a stop watch. The data, in the order collected, are given in Table 3.13, where the codes 1, 2, 3, 4 denote the colors pink, yellow, orange, blue, respectively.

Table 3.13 Times (in seconds) for the balloon experiment

Time order	1	2	3	4	5	6	7	8
Coded color	1	3	1	4	3	2	2	2
Inflation time	22.0	24.6	20.3	19.8	24.3	22.2	28.5	25.7
Time order	9	10	11	12	13	14	15	16
Coded color	3	1	2	4	4	4	3	1
Inflation time	20.2	19.6	28.8	24.0	17.1	19.3	24.2	15.8
Time order	17	18	19	20	21	22	23	24
Coded color	2	1	4	3	1	4	4	2
Inflation time	18.3	17.5	18.7	22.9	16.3	14.0	16.6	18.1
Time order	25	26	27	28	29	30	31	32
Coded color	2	4	2	3	3	1	1	3
Inflation time	18.9	16.0	20.1	22.5	16.0	19.3	15.9	20.3



Notes

---

---

---

---

---

---

---

---

---

---

Notes

---

---

---

---

---

---

---

---

---

---