Lecture 26

Time Series Analysis

STAT 8020 Statistical Methods II
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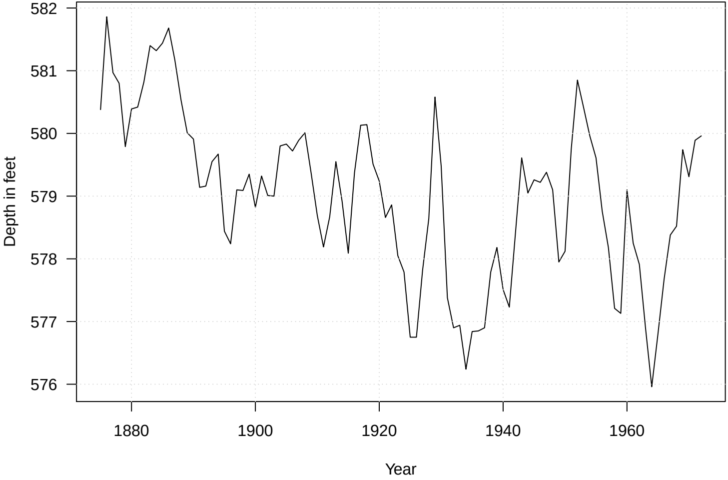
Agenda

1. Time Series Data
2. Objectives of Time Series Analysis
3. Features of Times Series
4. Means & Autocovariances
5. A Case Study

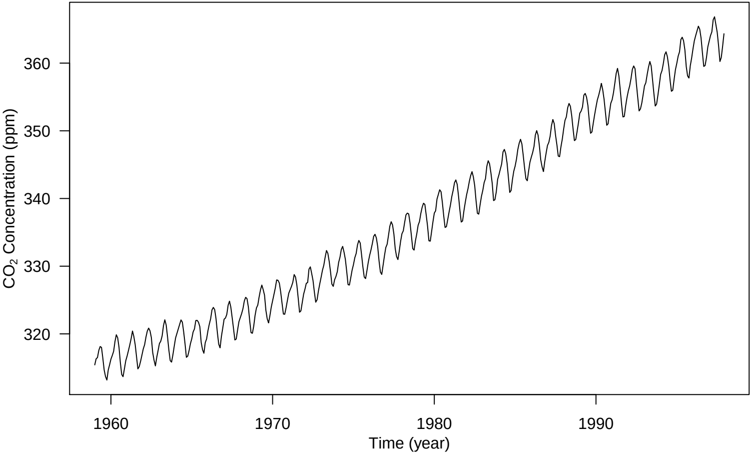
Level of Lake Huron 1875–1972

Annual measurements of the level of Lake Huron in feet.
[Source: Brockwell & Davis, 1991]

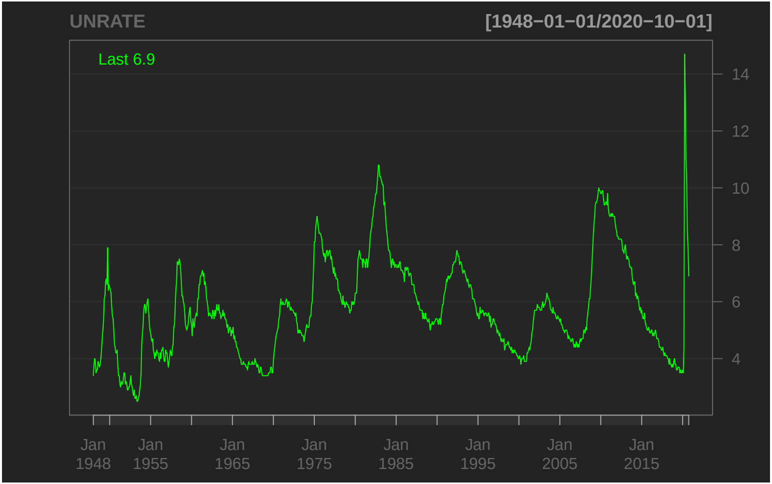


Mauna Loa Atmospheric $\mathrm{CO_2}$ Concentration

Monthly atmospheric concentrations of $\mathrm{CO_2}$ at the Mauna Loa Observatory [Source: Keeling & Whorf, Scripps Institution of Oceanography (SIO)]

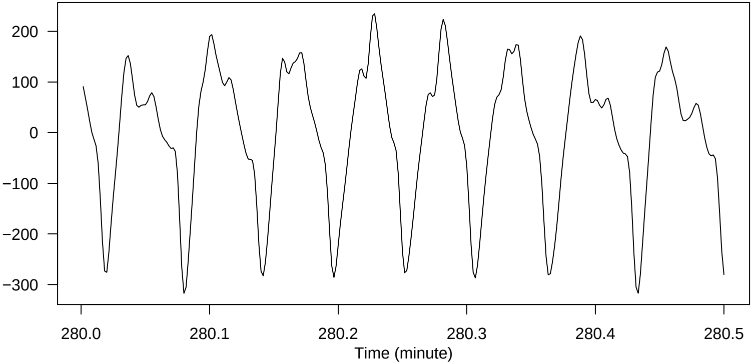


US Unemployment Rate 1948 Jan. – 2020 Oct.



Airflow Signal

A "normal" patient's 100 Hz sleep airflow signal [Source: Huang et al. 2020+]



Time Series Data & Models

- A time series is a set of observations made sequentially in time
- Time series analysis is the area of statistics which deals with the analysis of dependency between different observations in time series data
- A time series model is a probabilistic model that describes ways that the series data $\{y_t\}$ could have been generated
- More specifically, a time series model is usually a probability model for $\{Y_t : t \in T\}$, a collection of random variables indexed in time

Some Objectives of Time Series Analysis

- Find a statistical model that adequately explains the dependence observed in a time series
- To conduct statistical inferences, e.g., Is there evidence of a decreasing trend in the Lake Huron depths?
- To forecast future values of the time series based on those we have already observed

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  - The noise process, η_t, is the component that is neither trend nor seasonality
  - We will focus on finding plausible (typically stationary) statistical models for this process

Combining Trend μ_t, Seasonality s_t, and Noise η_t Together

There are two commonly used approaches

- Additive model:

$$y_t = \mu_t + s_t + \eta_t$$

- Multiplicative model:

$$y_t = \mu_t s_t \eta_t$$

  If all $\{y_t\}$ are positive then we obtain the additive model by taking logarithms:

$$\log y_t = \log \mu_t + \log s_t + \log \eta_t$$

Means, Autocovariances, and Stationary Processes

- The mean function of $\{Y_t\}$ is

$$\mu_t = \mathrm{E}[Y_t], \quad t \in T$$

- The autocovariance function of $\{Y_t\}$ is

$$\gamma(t, t') = \mathrm{Cov}(Y_t, Y_{t'}) = \mathrm{E}[(Y_t - \mu_t)(Y_{t'} - \mu_{t'})], \quad t, t' \in T$$

When $t = t'$ we obtain
$\gamma(t, t') = \mathrm{Cov}(Y_t, Y_t) = \mathrm{Var}(Y_t) = \sigma_t^2$, the variance function of Y_t

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Autocorrelation Function

The autocorrelation function (ACF) of $\{Y_t\}$ is

$$\rho(t,t') = \mathrm{Corr}(Y_t, Y_{t'}) = \frac{\gamma(t,t')}{\sqrt{\gamma(t,t)\gamma(t',t')}}$$

It measures the strength of linear association between Y_t and $Y_{t'}$

Properties:

1. $-1 \leq \rho(t,t') \leq 1, \quad t,t' \in T$
2. $\rho(t,t') = \rho(t',t), \quad \forall t,t' \in T$; $\rho(t,t) = 1, \quad \forall t \in T$
3. $\rho(t,t')$ is a non-negative definite function

Stationary Processes

We will still try to keep our models for $\{\eta_t\}$ as simple as possible by assuming stationarity, meaning that some characteristic of $\{\eta_t\}$ does not depend on the time points, only on the "time lag" between time points:

- $\mathrm{E}[\eta_t] = 0, \quad \forall t \in T$
- $\mathrm{Cov}(\eta_t, \eta_{t'}) = \gamma(t' - t) = \mathrm{Cov}(\eta_{t+s}, \eta_{t'+s})$

$\Rightarrow$ autocorrelation function (ACF):

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

Autoregressive Moving Average (ARMA) Models

Let $\{Z_t\}$ be independent and identical random variables that follow $\mathrm{N}(0, \sigma^2)$

- Moving Average Processes (MA(q)):
  $\eta_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} \cdots + \theta_q Z_{t-q}$

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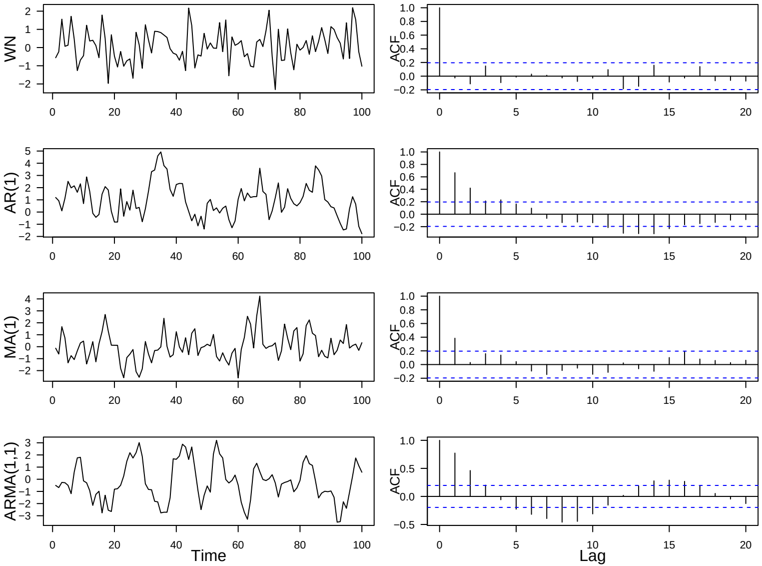
- Autoregressive Moving Average Processes ARMA(p,q):
  $\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + \cdots + \phi_p \eta_{t-p} + Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$

Autocorrelation Plot



Lake Huron Case Study

Source: `https://www.worldatlas.com/articles/what-states-border-lake-huron.html`

- Detrending
- Model selection and fitting
- Forecasting

See R lab 22 for a demo