Lecture 5
Multiple Linear Regression I
Reading: Chapter 12

STAT 8020 Statistical Methods II
September 3, 2020

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Agenda

1. Multiple Linear Regression
2. Estimation & Inference
3. General Linear Test
4. Multicollinearity

Multiple Linear Regression (MLR)

Goal: To model the relationship between two or more explanatory variables (X's) and a response variable (Y) by fitting a **linear equation** to observed data:

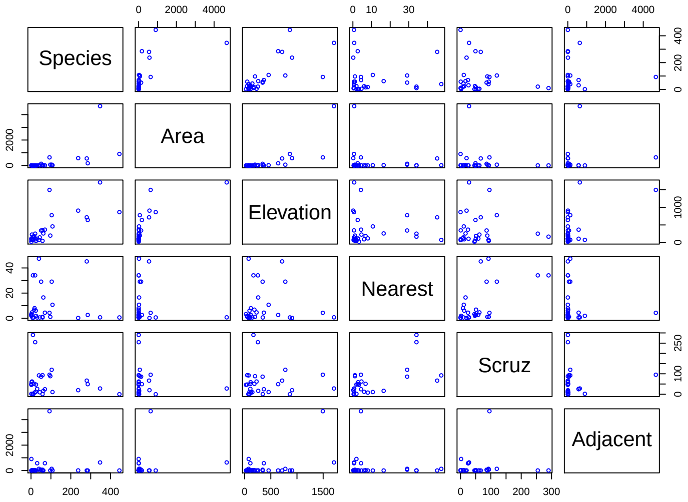
$$Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_{p-1} X_{p-1} + \varepsilon_i, \quad \varepsilon_i \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2)$$

Example: Species diversity on the Galapagos Islands. We are interested in studying the relationship between the number of plant species (`Species`) and the following geographic variables: `Area`, `Elevation`, `Nearest`, `Scruz`, `Adjacent`.

Data: Species Diversity on the Galapagos Islands

	Species	Endemics	Area	Elevation	Nearest	Scruz	Adjacent
Baltra	58	23	25.09	346	0.6	0.6	1.84
Bartolome	31	21	1.24	109	0.6	26.3	572.33
Caldwell	3	3	0.21	114	2.8	58.7	0.78
Champion	25	9	0.10	46	1.9	47.4	0.18
Coamano	2	1	0.05	77	1.9	1.9	903.82
Daphne.Major	18	11	0.34	119	8.0	8.0	1.84
Daphne.Minor	24	0	0.08	93	6.0	12.0	0.34
Darwin	10	7	2.33	168	34.1	290.2	2.85
Eden	8	4	0.03	71	0.4	0.4	17.95
Enderby	2	2	0.18	112	2.6	50.2	0.10
Espanola	97	26	58.27	198	1.1	88.3	0.57
Fernandina	93	35	634.49	1494	4.3	95.3	4669.32
Gardner1	58	17	0.57	49	1.1	93.1	58.27
Gardner2	5	4	0.78	227	4.6	62.2	0.21
Genovesa	40	19	17.35	76	47.4	92.2	129.49
Isabela	347	89	4669.32	1707	0.7	28.1	634.49
Marchena	51	23	129.49	343	29.1	85.9	59.56
Onslow	2	2	0.01	25	3.3	45.9	0.10
Pinta	104	37	59.56	777	29.1	119.6	129.49
Pinzon	108	33	17.95	458	10.7	10.7	0.03
Las.Plazas	12	9	0.23	94	0.5	0.6	25.09
Rabida	70	30	4.89	367	4.4	24.4	572.33
SanCristobal	280	65	551.62	716	45.2	66.6	0.57
SanSalvador	237	81	572.33	906	0.2	19.8	4.89
SantaCruz	444	95	903.82	864	0.6	0.0	0.52
SantaFe	62	28	24.08	259	16.5	16.5	0.52
SantaMaria	285	73	170.92	640	2.6	49.2	0.10
Seymour	44	16	1.84	147	0.6	9.6	25.09
Tortuga	16	8	1.24	186	6.8	50.9	17.95
Wolf	21	12	2.85	253	34.1	254.7	2.33

How Do Geographic Variables Affect Species Diversity?



Let's Take a Look at the Correlation Matrix

```
> round(cor(gala[, -2]), 3)
          Species   Area Elevation Nearest  Scruz Adjacent
Species     1.000  0.618     0.738  -0.014 -0.171    0.026
Area        0.618  1.000     0.754  -0.111 -0.101    0.180
Elevation   0.738  0.754     1.000  -0.011 -0.015    0.536
Nearest    -0.014 -0.111    -0.011   1.000  0.615   -0.116
Scruz      -0.171 -0.101    -0.015   0.615  1.000    0.052
Adjacent    0.026  0.180     0.536  -0.116  0.052    1.000
```

Model 1: Species ∼ Elevation

```
Call:
lm(formula = Species ~ Elevation, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max 
-218.319  -30.721  -14.690    4.634  259.180 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 11.33511   19.20529   0.590     0.56    
Elevation    0.20079    0.03465   5.795 3.18e-06 ***
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 78.66 on 28 degrees of freedom
Multiple R-squared:  0.5454,	Adjusted R-squared:  0.5291 
F-statistic: 33.59 on 1 and 28 DF,  p-value: 3.177e-06
```

Model 2: Species ∼ Elevation + Area

```
Call:
lm(formula = Species ~ Elevation + Area, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max
-192.619  -33.534  -19.199    7.541  261.514

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.10519   20.94211   0.817  0.42120
Elevation    0.17174    0.05317   3.230  0.00325 **
Area         0.01880    0.02594   0.725  0.47478
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 79.34 on 27 degrees of freedom
Multiple R-squared:  0.554,	Adjusted R-squared:  0.521
F-statistic: 16.77 on 2 and 27 DF,  p-value: 1.843e-05
```

Model 3: Species ~ Elevation + Area + Adjacent

```
Call:
lm(formula = Species ~ Elevation + Area + Adjacent, data = gala)

Residuals:
     Min       1Q   Median       3Q      Max
-124.064  -34.283   -8.733   27.972  195.973

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -5.71893   16.90706  -0.338  0.73789
Elevation    0.31498    0.05211   6.044  2.2e-06 ***
Area        -0.02031    0.02181  -0.931  0.36034
Adjacent    -0.07528    0.01698  -4.434  0.00015 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 61.01 on 26 degrees of freedom
Multiple R-squared:  0.746,     Adjusted R-squared:  0.7167
F-statistic: 25.46 on 3 and 26 DF,  p-value: 6.683e-08
```

“Full Model”

```
lm(formula = Species ~ Area + Elevation + Nearest + Scruz + Adjacent,
    data = gala)

Residuals:
     Min       1Q   Median       3Q      Max
-111.679  -34.898   -7.862   33.460  182.584

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  7.068221  19.154198   0.369 0.715351
Area        -0.023938   0.022422  -1.068 0.296318
Elevation    0.319465   0.053663   5.953 3.82e-06
Nearest      0.009144   1.054136   0.009 0.993151
Scruz       -0.240524   0.215402  -1.117 0.275208
Adjacent    -0.074805   0.017700  -4.226 0.000297

(Intercept)
Area
Elevation   ***
Nearest
Scruz
Adjacent    ***
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 60.98 on 24 degrees of freedom
Multiple R-squared:  0.7658,    Adjusted R-squared:  0.7171
F-statistic:  15.7 on 5 and 24 DF,  p-value: 6.838e-07
```

MLR Topics

Similar to SLR, we will discuss

- Estimation
- Inference
- Diagnostics and Remedies

We will also discuss some new topics

- Model Selection
- Multicollinearity

Multiple Linear Regression in Matrix Notation

Multiple Linear Regression (MLR):

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & X_{1,1} & X_{2,1} & \cdots & X_{p-1,1} \\ 1 & X_{1,2} & X_{2,2} & \cdots & X_{p-1,2} \\ \vdots & \ldots & \ddots & \vdots & \vdots \\ 1 & X_{1,n} & X_{2,n} & \cdots & X_{p-1,n} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

We can express MLR as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Error Sum of Squares (SSE) $= \sum_{i=1}^{n}(Y_i - \beta_0 - \sum_{j=1}^{p-1}\beta_j X_j)^2$ can be expressed in matrix notation as:

$$(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})$$

> Again, we are going to find $\hat{\beta}$ to minimize SSE as our estimate for $\boldsymbol{\beta}$

Estimation of Regression Coefficients

- The resulting least squares estimate is

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{Y}$$

- Fitted values:

$$\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{Y}$$

- Residuals:

$$\boldsymbol{e} = \boldsymbol{Y} - \hat{\boldsymbol{Y}} = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$$

Estimation of σ^2

- Similar approach as we did in SLR

$$\begin{aligned}\hat{\sigma}^2 &= \frac{\boldsymbol{e}^T\boldsymbol{e}}{n-p} \\ &= \frac{(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})^T(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-p} \\ &= \frac{\text{SSE}}{n-p} \\ &= \text{MSE}\end{aligned}$$

ANOVA Table

Source	df	SS	MS	F Value
Model	$p-1$	SSR	MSR = SSR/$(p-1)$	MSR/MSE
Error	$n-p$	SSE	MSE = SSE/(n-p)	
Total	$n-1$	SST		

- F Test: Tests if the predictors $\{X_1, \cdots, X_{p-1}\}$ collectively help explain the variation in Y
  - $H_0 : \beta_1 = \beta_2 = \cdots = \beta_{p-1} = 0$
  - $H_a :$ at least one $\beta_k \neq 0, \quad 1 \leq k \leq p-1$
  - $F^* = \frac{\text{MSR}}{\text{MSE}} = \frac{\text{SSR}/(p-1)}{\text{SSE}/(n-p)} \overset{H_0}{\sim} F(p-1, n-p)$
  - Reject H_0 if $F^* > F(1-\alpha, p-1, n-p)$

Testing Individual Predictor

- We can show that $\hat{\boldsymbol{\beta}} \sim \mathrm{N}_p\left(\boldsymbol{\beta}, \sigma^2\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right) \Rightarrow$ $\hat{\beta}_k \sim \mathrm{N}(\beta_k, \sigma^2_{\hat{\beta}_k})$
- Perform **t test**:
  - $H_0 : \beta_k = 0$ vs. $H_a : \beta_k \neq 0$
  - $\frac{\hat{\beta}_k - \beta_k}{\hat{\sigma}_{\hat{\beta}_k}} \sim t_{n-p} \Rightarrow t^* = \frac{\hat{\beta}_k}{\hat{\sigma}_{\hat{\beta}_k}} \overset{H_0}{\sim} t_{n-p}$
  - Reject H_0 if $|t^*| > t_{1-\alpha/2,n-p}$
- Confidence interval for β_k: $\hat{\beta}_k \pm t_{1-\alpha/2,n-p}\hat{\sigma}_{\hat{\beta}_k}$

Coefficient of Determination

- Coefficient of Determination R^2 describes proportional of the variance in the response variable that is predictable from the predictors

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SSR}}, \quad 0 \le R^2 \le 1$$

- R^2 usually increases with the increasing p, the number of the predictors
  - Adjusted R^2, denoted by $R^2_{\text{adj}} = \frac{\text{SSR}/(n-p)}{\text{SST}/(n-1)}$ attempts to account for p

R^2 vs. R^2_{adj} Example

Suppose the true relationship between response Y and predictors (X_1, X_2) is

$$Y = 5 + 2X_1 + \varepsilon,$$

where $\varepsilon \sim$ N(0, 1) and X_1 and X_2 are independent to each other. Let's fit the following two models to the "data"

Model 1: $Y = \beta_0 + \beta_1 X_1 + \varepsilon^1$
Model 2: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon^2$

Question: Which model will "win" in terms of R^2?

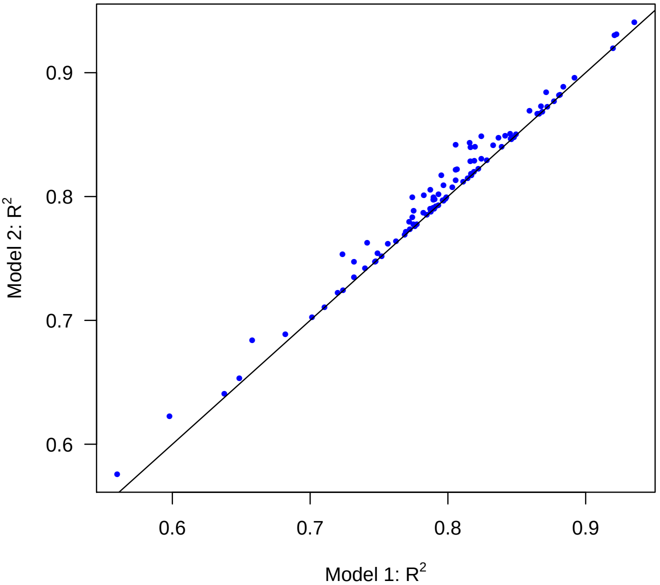
Model 1 Fit

```
> summary(fit1)

Call:
lm(formula = y ~ x1)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.6085 -0.5056 -0.2152  0.6932  2.0118 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   5.1720     0.1534   33.71  < 2e-16 ***
x1            1.8660     0.1589   11.74 2.47e-12 ***
---
Signif. codes:  
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8393 on 28 degrees of freedom
Multiple R-squared:  0.8313,	Adjusted R-squared:  0.8253 
F-statistic:   138 on 1 and 28 DF,  p-value: 2.467e-12
```

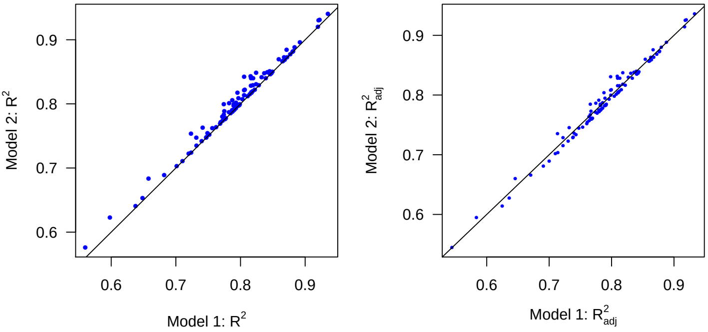
Model 2 Fit

```
> summary(fit2)

Call:
lm(formula = y ~ x1 + x2)

Residuals:
    Min      1Q  Median      3Q     Max
-1.3926 -0.5775 -0.1383  0.5229  1.8385

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.1792     0.1518  34.109  < 2e-16 ***
x1            1.8994     0.1593  11.923 2.88e-12 ***
x2           -0.2289     0.1797  -1.274    0.213
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8301 on 27 degrees of freedom
Multiple R-squared:  0.8408,    Adjusted R-squared:  0.8291
F-statistic: 71.32 on 2 and 27 DF,  p-value: 1.677e-11
```

R^2: Model 1 vs. Model 2



R^2_{adj}: Model 1 vs. Model 2



General Linear Test

- Comparison of a "full model" and "reduced model" that involves a subset of full model predictors
- Consider a full model with k predictors and reduced model with l predictors ($l < k$)
- Test statistic: $F^* = \frac{\text{SSE(R)} - \text{SSE}(F)/(k-1)}{\text{SSE}(F)/(n-k-1)}$ $\Rightarrow$ Testing H_0 that the regression coefficients for the extra variables are all zero

Species Diversity on the Galapagos Islands Revisited: Reduce Model

```
> summary(gala_fit1)

Call:
lm(formula = Species ~ Elevation)

Residuals:
     Min       1Q   Median       3Q      Max 
-218.319  -30.721  -14.690    4.634  259.180 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 11.33511   19.20529   0.590     0.56    
Elevation    0.20079    0.03465   5.795 3.18e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 78.66 on 28 degrees of freedom
Multiple R-squared:  0.5454,	Adjusted R-squared:  0.5291 
F-statistic: 33.59 on 1 and 28 DF,  p-value: 3.177e-06
```

Species Diversity on the Galapagos Islands Revisited: Full Model

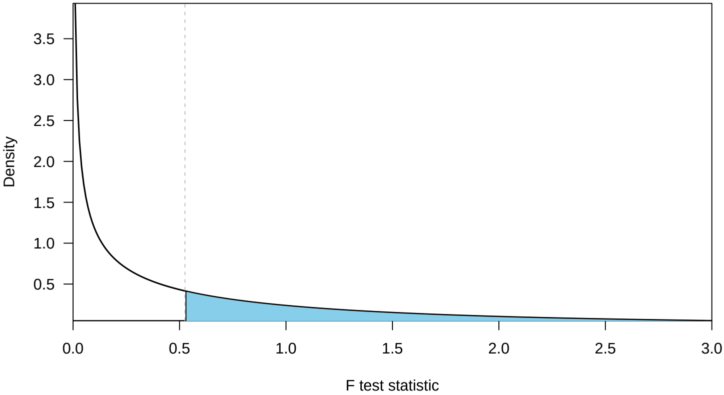
```
> summary(gala_fit2)

Call:
lm(formula = Species ~ Elevation + Area)

Residuals:
     Min       1Q   Median       3Q      Max 
-192.619  -33.534  -19.199    7.541  261.514 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)   
(Intercept) 17.10519   20.94211   0.817  0.42120   
Elevation    0.17174    0.05317   3.230  0.00325 **
Area         0.01880    0.02594   0.725  0.47478   
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 79.34 on 27 degrees of freedom
Multiple R-squared:  0.554,	Adjusted R-squared:  0.521 
F-statistic: 16.77 on 2 and 27 DF,  p-value: 1.843e-05
```

Perform a General Linear Test

- $H_0 : \beta_{\texttt{Area}} = 0$ vs. $H_a : \beta_{\texttt{Area}} \neq 0$
- $F^* = \frac{(173254-169947)/(2-1)}{169947/(30-2-1)} = 0.5254$
- P-value: $\mathrm{P}[F > 0.5254] = 0.4748$, where $F \sim \mathsf{F}(1, 27)$

```
> anova(gala_fit1, gala_fit2)
Analysis of Variance Table

Model 1: Species ~ Elevation
Model 2: Species ~ Elevation + Area
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     28 173254
2     27 169947  1      3307 0.5254 0.4748
```

P-value Calculation



P-value is the shaped area under the under the density curve

Multicollinearity

Another Simulated Example: Suppose the true relationship between response Y and predictors (X_1, X_2) is

$$Y = 4 + 0.8X_1 + 0.6X_2 + \varepsilon,$$

where $\varepsilon \sim \mathrm{N}(0, 1)$ and X_1 and X_2 are positively correlated with $\rho = 0.9$. Let's fit the following model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

```
Call:
lm(formula = Y ~ X1 + X2)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.63912 -0.59978  0.01897  0.58691  1.74518 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   4.0154     0.1646  24.390  < 2e-16 ***
X1           -0.1032     0.3426  -0.301    0.766    
X2            1.7471     0.3654   4.781 5.48e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8601 on 27 degrees of freedom
Multiple R-squared:  0.8166,    Adjusted R-squared:  0.803 
F-statistic: 60.12 on 2 and 27 DF,  p-value: 1.135e-10
```

Multicollinearity cont'd

- Numerical issue $\Rightarrow$ the matrix $\boldsymbol{X}^T\boldsymbol{X}$ is nearly singular
- Statistical issue
  - β's are not well estimated
  - β's may be meaningless
  - R^2 and predicted values are usually OK